[Quiz 2, Friday January 25th] Name:

Find the following integrals:

1. $\displaystyle\int_0^1 \frac{x^3}{\sqrt{1-x^2}}dx$ (using a substitution, $u=?$, $du=?$)

   Let $u = 1-x^2$, $du = -2xdx$, so that $x^3dx = x^2(xdx) = (1-u)\frac{du}{-2}$. We get (note the change in the limits of integration)

$$\int_0^1 \frac{x^3}{\sqrt{1-x^2}}dx = \int_1^0 \frac{(1-u)\frac{du}{-2}}{\sqrt{u}} = -\frac{1}{2}\int_1^0 (u^{-1/2} - u^{1/2})du = -\frac{1}{2}\left(2u^{1/2} - \frac{2}{3}u^{3/2}\right)\Big|_1^0 = \frac{2}{3}.$$

2. $\displaystyle\int_0^1 \frac{x^3}{\sqrt{1-x^2}}dx$ (using an inverse trigonometric substitution, $x=?$, $dx=?$)

   Letting $x = \sin\theta$, $dx = \cos\theta\ d\theta$, we get

$$\int_0^1 \frac{x^3}{\sqrt{1-x^2}}dx = \int_0^{\pi/2} \frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\ d\theta = \int_0^{\pi/2}\frac{\sin^3\theta}{\cos\theta}\cos\theta\ d\theta = \int_0^{\pi/2}\sin^3\theta\ d\theta.$$

   To integrate $\sin^3\theta$ we use the Pythagorean identity and a subsitution, $u=\cos\theta$, $du = -\sin\theta\ d\theta$ to get

$$\int_0^{\pi/2}\sin^3\theta\ d\theta = \int_0^{\pi/2}(1-\cos^2\theta)\sin\theta\ d\theta = -\int_1^0(1-u^2)du = -u + u^3/3\Big|_1^0 = 2/3.$$

   If we had been doing an indefinite integral, we would need to convert $u^3/3 - u$ back to x:

$$\begin{aligned} u^3/3 - u &= \frac{1}{3}\cos^3\theta - \cos\theta = \frac{1}{3}\cos^3(\arcsin x) - \cos(\arcsin x) \\ &= \frac{1}{3}(1-x^2)^{3/2} - (1-x^2)^{1/2} = -\frac{1}{3}(1-x^2)^{1/2}(x^2+2). \end{aligned}$$

   (Draw a triangle to see that $\cos(\arcsin x) = \sqrt{1-x^2}$).

3. $\displaystyle\int \frac{\ln y}{y^4}dy$

   Integrate by parts with

$$u = \ln y,\ dv = y^{-4}dy,\ du = \frac{dy}{y},\ v = -\frac{1}{3y^3},$$

   to get

$$\int \frac{\ln y}{y^4}dy = -\frac{\ln y}{3y^3} - \int -\frac{dy}{3y^4} = -\frac{\ln y}{3y^3} + \frac{1}{3}\int y^{-4}dy = -\frac{\ln y}{3y^3} - \frac{1}{9y^3} = -\frac{1}{3y^3}(\ln y + 1/3).$$

4. $\displaystyle\int \cos^2(\omega t)dt$ ($\omega \neq 0$ constant)

   Using the identity $\cos^2 x = \frac{1+\cos(2x)}{2}$, we have

$$\int \cos^2(\omega t)dt = \int \frac{1+\cos(2\omega t)}{2}dt = \frac{t}{2} + \frac{\sin(2\omega t)}{4\omega}.$$