MATH 2300-004 QUIZ 12 Name: ____________________________

Collaborators (if any):

Due Wednesday, May 1nd at the beginning of class. Submit your work on additional paper, treating this page as a cover sheet. You may use technology and work with with other students. If you work with others, please list their names above. The first two problems are essential, while the last two are extra-cirricular.

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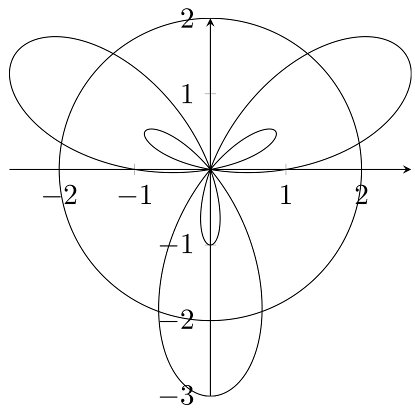
1. The polar curves

$$r(\theta) = 1 + 2\sin(3\theta),\ r = 2,$$

are graphed below.



(a) Find the area inside the larger loops and outside the smaller loops of the graph of $r = 1 + 2\sin(3\theta)$. [Hint: Use symmetry, the answer is $\pi + 3\sqrt{3}$.]

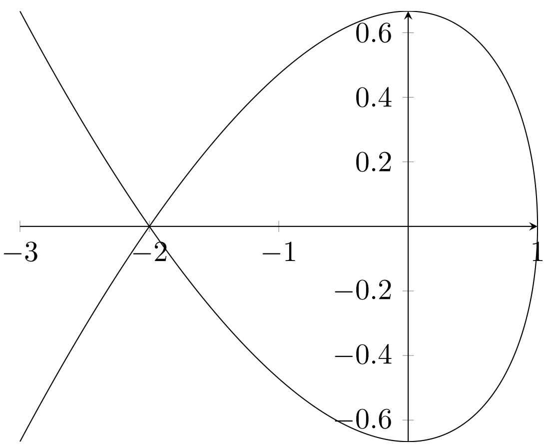
(b) Find the area outside the circle $r = 2$ but inside the curve $r = 1 + 2\sin(3\theta)$. [Answer: $\frac{5\sqrt{3}}{2} - \frac{\pi}{3}$.]

(c) What is the tangent line to the curve $r = 1 + 2\sin(3\theta)$ at the point in the first quadrant where r is maximum?

(d) Write down a definite integral for the arclength of the curve $r(\theta) = 1 + 2\sin(3\theta)$ and use a computer to evaluate. [Answer: $27.2667\ldots$.]

2. Consider the parametric curve defined by

$$x(t) = 1 - t^2, \; y(t) = t - t^3/3.$$



(a) Find the equations of the tangent lines to the curve at the point $(-2, 0)$.

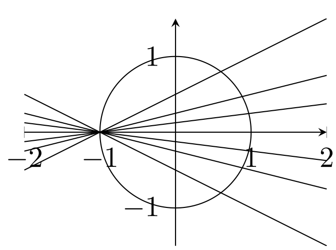
(b) When/where does the curve have horizontal tangents?

(c) What is the length of the part of the curve forming the "loop"? [Answer: $4\sqrt{3}$.]

3. Find the other point of intersection of the line with slope t going through the point $(-1, 0)$ and the circle of radius one,

$$y = t(x + 1), \; x^2 + y^2 = 1,$$

as a function of t. [This gives a rational parameterization of the unit circle, $(x(t), y(t)) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$.] Writing $t = m/n$, clear denominators in $x(t)^2 + y(t)^2 = 1$ to give a parameterization of Pythagorean triples, integers (a, b, c) with $a^2 + b^2 = c^2$.



4. This problem explores the relationship between polar coordinates and complex numbers.

- For $\theta \in \mathbb{R}$, use the power series for e^z to show that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where $i^2 = -1$ (at least formally, since we won't discuss convergence of series of complex numbers). [Hence polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ can be written in terms of complex numbers as

$$x + iy = r\cos\theta + ir\sin\theta = re^{i\theta},$$

with x and y are the real and imaginary part of $re^{i\theta}$.]

- Show that the exponential identity

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$$

is equivalent to the trigonometric identities

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi), \ \sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi).$$

[In particular, to multiply two complex numbers

$$z = x + iy = re^{i\theta}, \ w = u + iv = se^{i\phi},$$

we multiply their lengths r, s, and add their angles θ, ϕ:

$$zw = (xu - yv) + i(xv + yu) = rse^{i(\theta+\phi)}.$$