MATH 2300-004 QUIZ 11 [take-home portion] Name: ______________________

The following variation on the logistic equation models logistic growth with constant harvesting:

$$\frac{dP}{dt} = kP(1 - P/M) - c.$$

For this problem consider the specific instance

$$\frac{dP}{dt} = 0.08P(1 - P/1000) - 15,$$

modeling fish population in a pond where 15 fish per week are caught (time t in weeks).

1. What are the equilibrium solutions to the differential equation in part (i.e. what are the constant solutions)?

   Solution. The equilibrium solutions are the zeros of the right-hand side of the differential equation, $P = 750$, 250.

2. Find the general solution of the differential equation. [Integrate using partial fractions. You should get something equivalent to $P(t) = \dfrac{750 - 250Ce^{-t/25}}{1 - Ce^{-t/25}}$ where C is an arbitrary constant.]

   Solution. Separating variables, multiplying by a constant, and integrating gives

$$\int \frac{dP}{P^2 - 1000P + 187500} = -\int \frac{dt}{12500}.$$

   Partial fractions on the left-hand side gives

$$\frac{1}{P^2 - 1000P + 187500} = \frac{1}{(P - 750)(P - 250)} = \frac{1/500}{P - 750} + \frac{-1/500}{P - 250}.$$

   Integrating, we obtain

$$\ln\left|\frac{P - 750}{P - 250}\right| = -\frac{t}{25} + C.$$

   Exponentiating gives (different C)

$$\frac{P - 750}{P - 250} = Ce^{-t/25}$$

   Finally, solving for P gives the general solution

$$P(t) = \frac{750 - 250Ce^{-t/25}}{1 - Ce^{-t/25}}$$

3. Find and interpret the solutions with initial conditions $P(0) = 200, 300$.

Solution. We need to solve for C in the following

$$200 = \frac{750 - 250C}{1 - C}, \ 300 = \frac{750 - 250C}{1 - C}$$

obtaining $C = 11$ and $C = -9$ respectively. So the two particular solutions are

$$P(t) = \frac{750 - 2750e^{-t/25}}{1 - 11e^{-t/25}}, \ P(t) = \frac{750 + 2250e^{-t/25}}{1 + 9e^{-t/25}}.$$

In the first solution ($P(0) = 200$), the population reaches zero at $t = 25\ln(11/3) \approx 32.48$ weeks, i.e. fishing at a rate of 15 fish/week is unsustainable, while in the second solution ($P(0) = 300$), as $t \to \infty$ the population approaches 750 and the population can sustain this level of fishing.