MATH 2300-004 QUIZ 10 Name: ____________________

Due Monday, April 15th at the beginning of class.

1. Consider the power series
$$f(x) = \sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}(x-1)^n.$$

   (a) Find the interval of convergence of $f(x)$.

   Solution. The ratio test tells us that the the series converges for x such that
$$\lim_{n\to\infty} \left| \frac{2^{n+1}(x-1)^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{2^n(x-1)^n} \right| = 2|x-1| \lim_{n\to\infty} \sqrt{1+1/n} = 2|x-1| < 1,$$
   i.e. for $|x-1| < 1/2 = R$, and diverges for $|x-1| > 1/2 = R$. At the endpoint $x = 1 + 1/2$, we have
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$
   which diverges by the p-test or integral test, and at the endpoint $x = 1 - 1/2$, we have
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}},$$
   which converges by the alternating series test. Hence the interval of convergence is $[1/2, 3/2)$.

   (b) Differentiate f term-by-term and find the interval of convergence for the resulting power series.

   Solution. Differentiating term-by-term gives
$$f'(x) = \sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}} n(x-1)^{n-1} = \sum_{n=0}^{\infty} 2^{n+1}\sqrt{n+1}(x-1)^n$$
   for $|x-1| < 1/2$. However, the resulting power series diverges at $x = 1 \pm 1/2$ (the terms of the series do not approach zero) and the interval of convergence is $(1/2, 3/2)$.

   (c) Integrate f term-by-term and find the interval of convergence for the resulting power series.

   Solution. Integrating term-by-term gives
$$\int f(x)dx = \int \left(\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}(x-1)^n \right) dx = C + \sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}} \frac{(x-1)^{n+1}}{n+1}$$
$$= C + \sum_{n=2}^{\infty} \frac{2^{n-1}}{n\sqrt{n-1}}(x-1)^n.$$
   for $|x-1| < 1/2$ (the radius of convergence doesn't change when integrating term-by-term). However, the resulting series converges at both endpoints $x = 1 \pm 1/2$ since the series
$$\sum_{n=2}^{\infty} \frac{1}{2n\sqrt{n-1}}, \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{2n\sqrt{n-1}},$$

both converge, say by comparison to $\sum_n \frac{1}{n^{3/2}}$.

2. Find a power series representation (centered at zero) for

$$\frac{1}{(1+x^3)^2},$$

(perhaps starting with the geometric series).

Solution. We have

$$\frac{1}{(1-y)^2} = \frac{d}{dy}\left(\frac{1}{1-y}\right) = \frac{d}{dy}\left(\sum_{n=0}^{\infty} y^n\right) = \sum_{n=1}^{\infty} ny^{n-1},$$

so that

$$\frac{1}{(1+x^3)^2} = \frac{1}{(1-(-x^3))^2} = \sum_{n=1}^{\infty} n(-x^3)^{n-1} = \sum_{k=0}^{\infty}(k+1)(-1)^k x^{3k},$$

(re-indexing $k = n-1$ in the last equality).

3. Solve the following initial value problems (explicitly for y as a function of x).

(a) $y' + y^2 \sin x = 0,\ y(0) = -1/2$

Solution. Rearranging, we have

$$\frac{dy}{dx} = -y^2 \sin x,\ \frac{dy}{y^2} = -\sin x dx$$

so that

$$\begin{aligned} \int \frac{dy}{y^2} &= -\int \sin x dx \\ -\frac{1}{y} &= \cos x + C \\ y &= \frac{-1}{\cos x + C}. \end{aligned}$$

If $y(0) = -1/2$ then $C = 1$ and the solution to the initial value problem is

$$y(x) = \frac{-1}{1+\cos x}.$$

(b) $y' = \frac{x^2}{y(1+x^3)},\ y(0) = -1$

Solution. Separating variables gives

$$y\ dy = \frac{x^2}{1+x^3}dx.$$

Integrating, we obtain

$$\int y\ dy = \int \frac{x^2}{1+x^3} dx,$$
$$\frac{y^2}{2} = \frac{1}{3}\ln|1+x^3| + C,$$
$$y = \pm\sqrt{\frac{2}{3}\ln|1+x^3| + C}.$$

If $y(0) = -1$, then we must have $C = 1$ and the negative square root,

$$y(x) = -\sqrt{\frac{2}{3}\ln|1+x^3| + 1}$$

4. Suppose $y(x)$ is the solution to the initial value problem

$$y' = x^2 - y^2,\ y(0) = 1.$$

Use Euler's method (starting at $x = 0$ and with step size 0.1) to approximate $y(0.5)$.

Solution. The approximation is $y(0.5) \approx 0.674295419$. The relevant data are in the table below, where $y_{n+1} = y_n + (0.1)(x_n^2 - y_n^2)$:

n	x_n	y_n	$x_n^2 - y_n^2$
0	0	1	-1
1	0.1	0.9	-0.8
2	0.2	0.82	-0.6324
3	0.3	0.75676	-0.482685698
4	0.4	0.70849143	-0.34196106
5	0.5	0.674295419	

5. Use the third degree Taylor polynomial (centered at zero) for $f(x) = \ln(1+x)$ to estimate $\ln(2)$ and use Taylor's inequality to give bounds on the error.

Solution. The first four derivatives of $f(x) = \ln(1+x)$ are

$$f'(x) = \frac{1}{1+x},\ f''(x) = \frac{-1}{(1+x)^2},\ f'''(x) = \frac{2}{(1+x)^3},\ f^{(4)}(x) = \frac{-6}{(1+x)^4}.$$

The third degree Taylor polynomial centered at zero is

$$T_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}.$$

To use $T_3(x)$ to approximate $\ln(2)$ we take $x = 1$, $\ln(2) \approx T_3(1) = 1 - 1/2 + 1/3 = 5/6$. A bound for the absolute value of the fourth derivative $f^{(4)}(x)$ on the interval $[0,1]$ is

$$|f^{(4)}(x)| = \left|\frac{-6}{(1+x)^4}\right| \leq 6 = M$$

(since $6/(1+t)^4$ is decreasing on $[0,1]$) and Taylor's inequality states that

$$|\ln(2) - T_3(1)| = |R_n(1)| \leq \frac{M}{(3+1)!}|1-0|^{3+1} = \frac{1}{4}.$$

Hence

$$7/12 = 5/6 - 1/4 \leq \ln(2) \leq 5/6 + 1/4 = 13/12.$$

The next two problems are extra-credit. Points awarded for them will be added to your quiz score (although the maximum score is still only 10/10).

1. In this problem, you will show that Euler's method converges to an actual solution of the initial value problem below as you take smaller and smaller step sizes.

   (a) Use Euler's method to obtain an estimate $E_n(x)$ of the solution to

   $$y' = y, \; y(0) = 1,$$

   at x by breaking up the interval between 0 and x into n equal pieces.

   Solution. We have step size x/n so that after n steps, we reach x. The first few iterations are

   $$\begin{aligned} x_0 &= 0, & y_0 &= 1 \\ x_1 &= x/n, & y_1 &= 1 + x/n \\ x_2 &= 2x/n, & y_2 &= (1+x/n) + (1+x/n)x/n = (1+x/n)^2 \\ x_3 &= 3x/n, & y_3 &= (1+x/n)^2 + (1+x/n)^2 x/n = (1+x/n)^3 \\ &\dots & &\dots \\ x_n &= nx/n = x, & y_n &= (1+x/n)^n = E_n(x). \end{aligned}$$

   (b) Find the limit as n approaches infinity in your previous answer, i.e. find

   $$E(x) := \lim_{n\to\infty} E_n(x).$$

   Solution. We have

   $$E(x) := \lim_{n\to\infty} E_n(x) = \lim_{n\to\infty} (1 + x/n)^n = e^x,$$

   (taking logarithms and applying l'Hopital's rule for instance).

   (c) Show that the limit $E(x)$ above satisfies the initial value problem.

   Solution. $E(x) = e^x$ satisfies $E' = E$ and $E(0) = 1$ so it solves the initial value problem.

2. Solve the following initial value problem using power series

   $$y'' + y = 0, \; y(0) = 0, \; y'(0) = 1,$$

   i.e. assume $y = \sum_{n=0}^{\infty} c_n x^n$ is a solution (where the coefficients c_n are the unknowns!) and solve for the c_n recursively. Do you recognize your solution?

   Solution. If $y = \sum_{n=0}^{\infty} c_n x^n$, then

   $$y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n,$$

   and we are trying to solve

   $$0 = y'' + y = \sum_{n=2}^{\infty} (n+2)(n+1)c_{n+2}x^n + \sum_{n=0}^{\infty} c_n x^n = \sum_{n=2}^{\infty} [(n+2)(n+1)c_{n+2} + c_n]x^n,$$

subject to $y(0) = 0$, $y'(0) = 1$. For the above to hold, all of the coefficients $(n+2)(n+1)c_{n+2} + c_n$ must be zero, and the initial conditions give us $c_0 = 0$, $c_1 = 1$. Hence

$$
\begin{aligned}
0 &= (0+2)(0+1)c_{0+2} + c_0,\ c_2 = \frac{-c_0}{2\cdot 1} = 0,\\
0 &= (1+2)(1+1)c_{1+2} + c_1,\ c_3 = \frac{-c_1}{3\cdot 2} = \frac{-1}{3!},\\
0 &= (2+2)(2+1)c_{2+2} + c_2,\ c_4 = \frac{-c_2}{4\cdot 3} = 0,\\
0 &= (3+2)(3+1)c_{3+2} + c_3,\ c_5 = \frac{-c_3}{5\cdot 4} = \frac{1}{5!},\\
0 &= (4+2)(4+1)c_{4+2} + c_4,\ c_6 = \frac{-c_4}{6\cdot 5} = 0,\\
&\dots,\\
0 &= (n+2)(n+1)c_{n+2} + c_n,\ c_{n+2} = \frac{-c_n}{(n+2)(n+1)} = 0 \text{ or } \frac{(-1)^n}{n!}.
\end{aligned}
$$

The resulting series is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ which we recognize as $\sin x$.

More generally, the relation $c_{n+2} = \dfrac{-c_n}{(n+2)(n+1)}$ give us

$$
c_{2k} = \frac{(-1)^k c_0}{(2k)!},\ c_{2k+1} = \frac{(-1)^k c_1}{(2k+1)!},
$$

so that the general solution to $y'' + y = 0$ is

$$
y(x) = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + c_1 \frac{(-1)^k x^{2k+1}}{(2k+1)!} = c_0 \cos x + c_1 \sin x.
$$