Integral test

Suppose $(a_n)_{n=1}^{\infty} = (f(n))_{n=1}^{\infty}$ where

- f is positive on $[1, \infty)$,
- f is decreasing on $[1, \infty)$,
- f is integrable on each interval $[1, T)$.

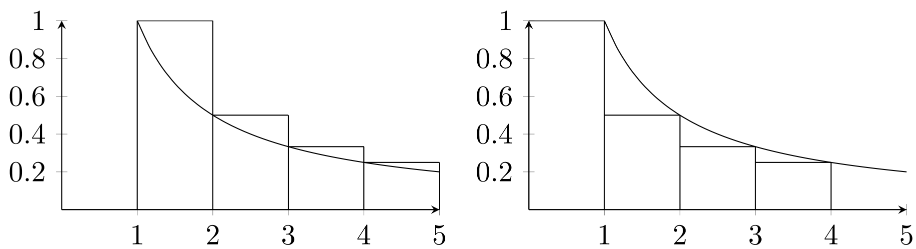
Then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\displaystyle\int_1^{\infty} f(x)dx$ converges. [In other words if one converges/diverges, so does the other.] Moreover, we have the following estimate on the remainder $R_N = \displaystyle\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$

$$\int_{N+1}^{\infty} f(x)dx \le R_N \le \int_N^{\infty} f(x)dx.$$

The series $\displaystyle\sum_{n=1}^{\infty} a_n$ can be interpreted as an improper integral of either of the piecewise constant functions

$$U(x) = a_n,\ n < x \le n+1,\ L(x) = a_n,\ n-1 < x \le n,$$

whose graphs are above and below the graph of $f(x)$ respectively.



You can apply the integral test to *tails* of series, i.e. forget finitely many initial terms, so that the sequence of terms need only be *eventually* decreasing and *eventually* positive.

For example, consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$. For $p \le 0$ the terms do not approach zero, so the series diverges. For $p > 0$, we can compare the series to the improper integral

$$\int_1^{\infty} \frac{dx}{x^p}$$

since $f(x) = 1/x^p$ is decreasing, positive, and continuous on $[1, \infty)$. For $p \ne 1$ We have

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{T\to\infty} \int_1^T \frac{dx}{x^p} = \lim_{T\to\infty} \frac{1}{(1-p)x^{p-1}}\Big|_1^T = \lim_{T\to\infty} \frac{1}{(1-p)T^{p-1}} - \frac{1}{1-p} = \begin{cases} \frac{1}{p-1} & p > 1 \\ \infty & 0 < p < 1 \end{cases},$$

while for $p = 1$

$$\int_1^{\infty} \frac{dx}{x} = \lim_{T\to\infty} \ln T = \infty.$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & p > 1 \\ \text{diverges} & p \le 1 \end{cases}.$$

Use the integral test to show the convergence or divergence of the following series. Make sure the hypotheses are satisfied:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

2. $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$

3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

1. For the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, find N such that the remainder R_N is less than $1/1000$.

2. For the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$, estimate the remainder after adding the first 10 terms.