Integration by substitution (chain rule backwards)

If g is differentiable at x and f is differentiable at $g(x)$, then $f(g(x))$ is differentiable with

$$[f(g(x))]' = f'(g(x))g'(x) \text{ (chain rule).}$$

Reversing this, if we are trying to integrate something of the form

$$\int f(u(x))\frac{du}{dx}dx$$

and we know an antiderivative for f, $F' = f$, then

$$\int f(u(x))\frac{du}{dx}dx = F(u(x)),$$

since the derivative of the right-hand side is the integrand of the left-hand side. In other words, we integrate by "substituting" the function $u(x)$ with the variable u and the differential $\frac{du}{dx}dx$ by du:

$$\int f(u(x))\frac{du}{dx}dx = \int f(u)du = F(u) = F(u(x)).$$

For definite integrals, one needs to change the limits of integration when making a substitution:

$$\int_{x=a}^{x=b} f(u(x))\frac{du}{dx}dx = F(u(x))\Big|_{x=a}^{x=b} = F(u(b)) - F(u(a)) = F(u)\Big|_{u=u(a)}^{u=u(b)} = \int_{u=u(a)}^{u=u(b)} f(u)du.$$

Some examples:

- $\displaystyle\int_0^{\pi/4} \tan x\ dx = \int_0^{\pi/4} \frac{\sin x}{\cos x}dx$. With

  $$u = \cos x,\ du = -\sin x dx,\ u(0) = 1,\ u(\pi/4) = 1/\sqrt{2},$$

  we have

  $$\int_0^{\pi/4} \frac{\sin x}{\cos x}dx = \int_1^{1/\sqrt{2}} \frac{du}{u} = \ln|u|\Big|_1^{1/\sqrt{2}} = \frac{\ln 2}{2}.$$

- $\displaystyle\int \frac{\cos(\pi/x)}{x^2}dx$. With $u = \pi/x$, $du = -\frac{\pi}{x^2}dx$, we have

  $$\int \frac{\cos(\pi/x)}{x^2}dx = -\frac{1}{\pi}\int \cos(\pi/x)\left(-\frac{\pi}{x^2}\right)dx = -\frac{1}{\pi}\int \cos u\ du = -\frac{\sin u}{\pi} = -\frac{\sin(\pi/x)}{\pi}.$$

- $\displaystyle\int_0^1 \frac{x}{1+x^4}dx = \int_0^1 \frac{x\ dx}{1+(x^2)^2}$. With $u = x^2$, $du = 2xdx$, $u(0) = 0$, $u(1) = 1$

  $$\frac{1}{2}\int_0^1 \frac{du}{1+u^2} = \frac{1}{2}\arctan u\Big|_0^1 = \frac{1}{2}(\pi/4 - 0) = \pi/8.$$

Try it out!

1. $\int x\sin(x^2)dx$

   With $u = x^2$, $du = 2xdx$, we have

$$\int x\sin(x^2)dx = \frac{1}{2}\int \sin u\ du = -\frac{1}{2}\cos u = -\frac{1}{2}\cos(x^2).$$

2. $\int \sqrt{x}(x+3)dx,\ \int x\sqrt{x+3}dx$

   The first requires no substitution

$$\int \sqrt{x}(x+3)dx = \int x^{3/2}dx + 3\int x^{1/2}dx = \frac{2}{5}x^{5/2} + 2x^{3/2}.$$

   In the second, with $u = x+3$, $x = u-3$, $du = dx$, we have

$$\int x\sqrt{x+3}dx = \int (u-3)\sqrt{u}\ du = \int u^2\ du - 3\int \sqrt{u}du = u^3/3 - 2u^{3/2}/3.$$

3. $\int_1^e \frac{\sqrt{\ln x}}{x}dx$

   With

$$u = \ln x,\ du = dx/x,\ u(e) = 1,\ u(1) = 0,$$

   we have

$$\int_1^e \frac{\sqrt{\ln x}}{x}dx = \int_0^1 \sqrt{u}\ du = \frac{2}{3}u^{3/2}\Big|_0^1 = 2/3.$$

4. $\int \frac{x+4}{x}dx,\ \int \frac{x}{x+4}dx$

   The first requires no substitution

$$\int \frac{x+4}{x}dx = \int dx + 4\int \frac{dx}{x} = 1 + 4\ln|x|.$$

   In the second, with $u = x+4$, $x = u-4$, $dx = du$, we have

$$\int \frac{x}{x+4}dx = \int \frac{u-4}{u}du = \int du - 4\int \frac{du}{u} = u - 4\ln|u| = x + 4 - 4\ln|x+4|.$$

5. $\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$

   With $u = e^x$, $du = e^x dx$, we have

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u = \arcsin(e^x).$$

6. $\int \frac{\arctan x}{1+x^2}dx$

   With $u = \arctan x$, $du = \frac{dx}{1+x^2}$, we have

$$\int \frac{\arctan x}{1+x^2}dx = \int u\ du = u^2/2 = \frac{1}{2}(\arctan x)^2.$$

7. $\displaystyle\int \frac{x^3}{(1+x^2)^2}dx$

With $u = 1+x^2$, $du = 2xdx$, we have $x^3dx = \dfrac{u-1}{2}du$ and

$$\int \frac{x^3}{(1+x^2)^2}dx = \int \frac{u-1}{2u^2}du = \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} = \frac{\ln|u|}{2} + \frac{1}{2u} = \frac{\ln|1+x^2|}{2} + \frac{1}{2(1+x^2)}.$$

8. $\displaystyle\int_{-2}^{0} \frac{x^2}{\sqrt{1-x^3}}dx$

With

$$u = 1-x^3,\ du = -3x^2dx,\ u(-2) = 9,\ u(0) = 1,$$

we have

$$\int_{-2}^{0} \frac{x^2}{\sqrt{1-x^3}}dx = -\frac{1}{3}\int_{9}^{1} \frac{du}{\sqrt{u}} = -\frac{2}{3}u^{1/2}\Big|_{9}^{1} = 4/3.$$