Integration by substitution (chain rule backwards)

If g is differentiable at x and f is differentiable at $g(x)$, then $f(g(x))$ is differentiable with

$$[f(g(x))]' = f'(g(x))g'(x) \text{ (chain rule).}$$

Reversing this, if we are trying to integrate something of the form

$$\int f(u(x))\frac{du}{dx}dx$$

and we know an antiderivative for f, $F' = f$, then

$$\int f(u(x))\frac{du}{dx}dx = F(u(x)),$$

since the derivative of the right-hand side is the integrand of the left-hand side. In other words, we integrate by "substituting" the function $u(x)$ with the variable u and the differential $\frac{du}{dx}dx$ by du:

$$\int f(u(x))\frac{du}{dx}dx = \int f(u)du = F(u) = F(u(x)).$$

For definite integrals, one needs to change the limits of integration when making a substitution:

$$\int_{x=a}^{x=b} f(u(x))\frac{du}{dx}dx = F(u(x))\Big|_{x=a}^{x=b} = F(u(b)) - F(u(a)) = F(u)\Big|_{u=u(a)}^{u=u(b)} = \int_{u=u(a)}^{u=u(b)} f(u)du.$$

Some examples:

- $\displaystyle\int_0^{\pi/4} \tan x \; dx = \int_0^{\pi/4} \frac{\sin x}{\cos x}dx$. With

  $$u = \cos x,\;\; du = -\sin x dx,\;\; u(0) = 1,\;\; u(\pi/4) = 1/\sqrt{2},$$

  we have

  $$\int_0^{\pi/4} \frac{\sin x}{\cos x}dx = \int_1^{1/\sqrt{2}} \frac{du}{u} = \ln|u|\Big|_1^{1/\sqrt{2}} = \frac{\ln 2}{2}.$$

- $\displaystyle\int \frac{\cos(\pi/x)}{x^2}dx$. With $u = \pi/x$, $du = -\frac{\pi}{x^2}dx$, we have

  $$\int \frac{\cos(\pi/x)}{x^2}dx = -\frac{1}{\pi}\int \cos(\pi/x)\left(-\frac{\pi}{x^2}\right)dx = -\frac{1}{\pi}\int \cos u \; du = -\frac{\sin u}{\pi} = -\frac{\sin(\pi/x)}{\pi}.$$

- $\displaystyle\int_0^1 \frac{x}{1+x^4}dx = \int_0^1 \frac{x\;dx}{1+(x^2)^2}$. With $u = x^2$, $du = 2xdx$, $u(0) = 0$, $u(1) = 1$

  $$\frac{1}{2}\int_0^1 \frac{du}{1+u^2} = \frac{1}{2}\arctan u\Big|_0^1 = \frac{1}{2}(\pi/4 - 0) = \pi/8.$$

Try it out!

1. $\int x\sin(x^2)dx$

2. $\int \sqrt{x}(x+3)dx, \ \int x\sqrt{x+3}dx$

3. $\int_1^e \frac{\sqrt{\ln x}}{x}dx$

4. $\int \frac{x+4}{x}dx, \ \int \frac{x}{x+4}dx$

5. $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$

6. $\displaystyle\int \frac{\arctan x}{1+x^2}dx$

7. $\displaystyle\int \frac{x^3}{(1+x^2)^2}dx$

8. $\displaystyle\int_{-2}^{0} \frac{x^2}{\sqrt{1-x^3}}dx$