

Integration by parts (product rule backwards)

The product rule states

$$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + f'(x)g(x).$$

Integrating both sides gives

$$f(x)g(x) = \int f(x)g'(x)dx + \int f'(x)g(x)dx.$$

Letting $f(x) = u$, $g(x) = v$, and rearranging, we obtain

$$\int u dv = uv - \int v du.$$

Some examples:

- $\int xe^x dx$. Differentiating $u = x$ gives a “simpler” function, $du = dx$, while integrating $dv = e^x dx$ gives $v = e^x$ which is no more difficult to integrate. Applying integration by parts gives

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x = e^x(x - 1).$$

- $\int \ln x dx$. Differentiating $u = \ln x$ gives the “simpler” function $du = \frac{1}{x}dx$ while integrating $dv = dx$ gives $v = x$. Applying integration by parts, we obtain

$$\int \ln x dx = x \ln x - \int x \frac{1}{x} dx = x \ln x - x = x(\ln x - 1).$$

- $\int e^x \sin x dx$. We can integrate by parts twice and solve algebraically for the integral as follows:

$$\begin{aligned} I &= \int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx = e^x \sin x - \left(e^x \cos x - \int e^x (-\sin x) dx \right) \\ &= e^x(\sin x - \cos x) - \int e^x \sin x dx = e^x(\sin x - \cos x) - I, \end{aligned}$$

$$2I = e^x(\sin x - \cos x),$$

$$I = \frac{e^x}{2}(\sin x - \cos x).$$

Some for you to try:

1. $\int \frac{\ln y}{\sqrt{y}} dy$
With

$$\begin{aligned} u &= \ln y & dv &= y^{-1/2} dy \\ du &= dy/y & v &= 2y^{1/2} \end{aligned}$$

we have

$$\int \frac{\ln y}{\sqrt{y}} dy = 2y^{1/2} \ln y - 2 \int y^{-1/2} dy = 2y^{1/2} \ln y - 4y^{1/2} = 2\sqrt{y}(\ln y - 2).$$

2. $\int \frac{x}{e^{2x}} dx$
With

$$\begin{aligned} u &= x & dv &= e^{-2x} dx \\ du &= dx & v &= -e^{-2x}/2 \end{aligned}$$

we have

$$\int \frac{x}{e^{2x}} dx = \int x e^{-2x} dx = -\frac{x e^{-2x}}{2} + \frac{1}{2} \int e^{-2x} dx = -\frac{x e^{-2x}}{2} - \frac{e^{-2x}}{4} = -\frac{e^{-2x}}{2} (x + 1/2).$$

3. $\int x \ln x dx$
With

$$\begin{aligned} u &= \ln x & dv &= x dx \\ du &= dx/x & v &= x^2/2 \end{aligned}$$

we have

$$\int x \ln x dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} = \frac{x^2}{2} (\ln x - 1/2)$$

4. $\int_0^1 \arccos z dz$
With

$$\begin{aligned} u &= \arccos z & dv &= dz \\ du &= \frac{-dz}{\sqrt{1-z^2}} & v &= z \end{aligned}$$

we have

$$\int_0^1 \arccos z dz = z \arccos z \Big|_0^1 + \int_0^1 \frac{z}{\sqrt{1-z^2}} dz.$$

We now make the substitution $u = 1 - z^2$, $du = -2z dz$ to obtain

$$\int_0^1 \frac{z}{\sqrt{1-z^2}} dz = -\frac{1}{2} \int_1^0 u^{-1/2} du = -\sqrt{u} \Big|_1^0 = 1.$$

Hence

$$\int_0^1 \arccos z dz = z \arccos z \Big|_0^1 + \int_0^1 \frac{z}{\sqrt{1-z^2}} dz = 0 - 0 + 1 = 1.$$

5. $\int x^3 \sqrt{1+x^2} dx$

You don't need integration by parts. With the substitution

$$u = 1 + x^2, \quad du = 2x dx, \quad x^2 = u - 1,$$

we have

$$\begin{aligned} \int x^3 \sqrt{1+x^2} dx &= \frac{1}{2} \int \frac{(u-1)du}{\sqrt{u}} = \frac{1}{2} \int (u^{1/2} - u^{-1/2}) du = \frac{1}{3} u^{3/2} - u^{1/2} = (u/3 - 1)\sqrt{u} \\ &= \frac{1}{3}(x^2 - 2)\sqrt{1+x^2}. \end{aligned}$$

However, you could use integration parts with

$$\begin{aligned} u &= x^2 & dv &= \frac{x}{\sqrt{1+x^2}} \\ du &= 2x dx & v &= \sqrt{1+x^2} \end{aligned}$$

(using a substitution to find $\int dv$). We get (with $w = 1 + x^2$, $dw = 2x dx$)

$$\begin{aligned} \int x^3 \sqrt{1+x^2} dx &= x^2 \sqrt{1+x^2} - \int 2x \sqrt{1+x^2} = x^2 \sqrt{1+x^2} - \int \sqrt{w} dw = x^2 \sqrt{1+x^2} - \frac{2}{3} w^{3/2} \\ &= x^2 \sqrt{1+x^2} - \frac{2}{3}(1+x^2)^{3/2} = \frac{1}{3}(x^2 - 2)\sqrt{1+x^2}. \end{aligned}$$

6. $\int_1^{\sqrt{3}} \arctan(1/x) dx$
With

$$\begin{aligned} u &= \arctan(1/x) & dv &= dx \\ du &= \frac{-dx}{x^2+1} & v &= x \end{aligned}$$

we get

$$\int_1^{\sqrt{3}} \arctan(1/x) dx = x \arctan(1/x) \Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x dx}{1+x^2}.$$

With $w = 1 + x^2$, $dw = 2x dx$, the resulting integral is

$$\int_1^{\sqrt{3}} \frac{x dx}{1+x^2} = \frac{1}{2} \int_2^4 \frac{dw}{w} = \ln 4 - \ln 2 = \ln 2,$$

so that the end result is

$$\int_1^{\sqrt{3}} \arctan(1/x) dx = x \arctan(1/x) \Big|_1^{\sqrt{3}} + \ln 2 = \sqrt{3} \frac{\pi}{6} - \frac{\pi}{4} + \ln 2 = \frac{(2\sqrt{3} - 3)\pi}{12} + \ln 2.$$

7. $\int (\ln x)^2 dx$
 With

$$\begin{aligned} u &= (\ln x)^2 & dv &= dx \\ du &= \frac{2 \ln x}{x} dx & v &= x \end{aligned}$$

we get

$$\int (\ln x)^2 dx = x(\ln x)^2 - 2 \int \ln x dx = x(\ln x)^2 - (x \ln x - x) = x(\ln x)^2 - x \ln x + x.$$

8. $\int t^2 \sin(\pi t) dt$

We will use integration by parts twice to get rid of the t^2 . With

$$\begin{aligned} u &= t^2 & dv &= \sin(\pi t) dt \\ du &= 2t dt & v &= -\cos(\pi t)/\pi \end{aligned}$$

we get

$$\int t^2 \sin(\pi t) dt = \frac{-t^2}{\pi} \cos(\pi t) + \frac{2}{\pi} \int t \cos(\pi t) dt.$$

Integration by parts on the resulting integral with

$$\begin{aligned} u &= t & dv &= \cos(\pi t) dt \\ du &= dt & v &= \sin(\pi t)/\pi \end{aligned}$$

gives

$$\int t \cos(\pi t) dt = \frac{t}{\pi} \sin(\pi t) - \frac{1}{\pi} \int \sin(\pi t) dt = \frac{t}{\pi} \sin(\pi t) + \frac{\cos(\pi t)}{\pi^2}.$$

Hence

$$\int t^2 \sin(\pi t) dt = \frac{-t^2}{\pi} \cos(\pi t) + \frac{2t}{\pi^2} \sin(\pi t) + \frac{2 \cos(\pi t)}{\pi^3}.$$

9. $\int e^{-x} \cos(\pi x) dx$

This is similar to the third example (on the first page).

10. $\int \cos(\sqrt{x}) dx$ (Make a substitution first.)

With the substitution $y = \sqrt{x}$, $dy = \frac{dx}{2\sqrt{x}}$, the integral becomes

$$\int \cos(\sqrt{x}) dx = 2 \int y \cos(y) dy.$$

Now use integration by parts with

$$\begin{aligned} u &= y & dv &= \cos y dy \\ du &= dy & v &= \sin y. \end{aligned}$$

11. $\int \sin(\ln w)dw$ (Make a substitution first.)

With the substitution $x = \ln w$, $dx = dw/w$, the integral becomes

$$\int \sin(\ln w)dw = \int e^x \sin x dx.$$

Now integrate as in the third example (on the first page).

12. $\int \sec^3 \theta d\theta$

With

$$\begin{aligned} u &= \sec \theta & dv &= \sec^2 \theta d\theta \\ du &= \sec \theta \tan \theta & v &= \tan \theta \end{aligned}$$

the integral becomes

$$\int \sec^3 \theta d\theta = \sec \theta \tan \theta - \int \tan^2 \theta \sec \theta d\theta.$$

Writing $\tan^2 \theta = \sec^2 \theta - 1$ gives

$$\int \tan^2 \theta \sec \theta d\theta = \int (\sec^2 \theta - 1) \sec \theta d\theta = \int \sec^3 \theta d\theta - \int \sec \theta d\theta = \int \sec^3 \theta d\theta - \ln |\sec \theta + \tan \theta|.$$

Hence

$$I = \int \sec^3 \theta d\theta = \sec \theta \tan \theta - \int \sec^3 \theta d\theta + \ln |\sec \theta + \tan \theta|,$$

$$2I = \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|,$$

$$I = \frac{1}{2}(\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|).$$