Euler's method

Suppose we want to approximate a solution to the initial value problem

$$y' = f(x, y),\ \ y(x_0) = y_0.$$

The slope of the tangent line to the actual solution is given by $f(x_0, y_0)$. If we go along the tangent line for a short time, say from x_0 to $x_0 + h$, we move from the point (x_0, y_0) to the new point

$$\begin{aligned} x_1 &= x_0 + h, \\ y_1 &= y_0 + f(x_0, y_0)h. \end{aligned}$$

If we repeat this process, following the tangent line to an actual solution through the point (x_1, y_1), which has slope $f(x_1, y_1)$, for another step of length h, we end up at the point

$$\begin{aligned} x_2 &= x_1 + h, \\ y_2 &= y_1 + f(x_1, y_1)h. \end{aligned}$$

In general, taking one more step of size h gives a new position (x_{n+1}, y_{n+1}) depending on the previous position (x_n, y_n) as follows:

$$\begin{aligned} x_{n+1} &= x_n + h, \\ y_{n+1} &= y_n + f(x_n, y_n)h. \end{aligned}$$

Problems

1. Fill in the table using 6 steps of size $h = 1/2$ to approximate the solution to

$$\frac{dy}{dx} = f(x_n, y_n) = xy,\ \ y(0) = 1,$$

at $x = 3$.

n	0	1	2	3	4	5	6
x_n	0						
y_n	1						
$f(x_n, y_n)$							

2. (a) It is easy to check that $y = e^x$ is a solution to the initial value problem

$$y' = y, \; y(0) = 1.$$

Break the interval $[0, 1]$ into 6 piece of equal length and use Euler's method to approximate e.

(b) (Challenge) Break the interval $[0, 1]$ into n pieces of equal length and use Euler's method to approximate e. Take the limit as $n \to \infty$ and show that your approximation actually converges to e.