## Error bound for the trapezoidal rule

We want to control the error

$$E_T(n) = \int_a^b f(x)dx - T_n,\ T_n = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} h$$

where $T_n$ is the sum of the areas of the trapezoids of equal width $h = \frac{b-a}{n}$ and parallel heights $f(x_i)$, $(x_{i+1})$, and $x_i = a + bh$ define the partition of $[a, b]$ into $n$ equal parts.

Let's consider one piece at a time. For the integral we integrate by parts twice, leaving arbitrary constants $C$ and $D$ when taking antiderivatives:

$$\int_{x_i}^{x_{i+1}} f(x)dx = \int_0^h f(t + x_i)dt$$

$$= [(t + C)f(t + x_i)]_0^h - \left[\left(\frac{(t + C)^2}{2} + D\right) f'(x_i + t)\right]_0^h + \int_0^h \left(\frac{(t + C)^2}{2} + D\right) f''(x_i + t)dt.$$

First, we choose $C$ so that the first term above is the area of the trapezoid in our approximation:

$$h\frac{f(x_i) + f(x_{i+1})}{2} = [(t + C)f(t + x_i)]_0^h = (C + h)f(x_{i+1}) - Cf(x_i),\ C = -\frac{h}{2}.$$

Next we choose $D$ so that the second term is zero:

$$0 = \left[\left(\frac{(t + C)^2}{2} + D\right) f'(x_i + t)\right]_0^h = \left(\frac{h^2}{8} + D\right) f'(x_{i+1}) - \left(\frac{h^2}{8} + D\right) f'(x_i),\ D = -\frac{h^2}{2}.$$

Hence we get

$$\int_{x_i}^{x_{i+1}} f(x)dx = \text{ area of trapezoid } + \int_0^h \left(\frac{(t - h/2)^2}{2} - \frac{h^2}{8}\right) f''(x_i + t)dt.$$

To bound the error, we have to bound the integral on the right. Assuming $|f''(x)| \leq K$ on the interval $[a, b]$, we can estimate

$$\left|\int_0^h \left(\frac{(t - h/2)^2}{2} - \frac{h^2}{8}\right) f''(x_i + t)dt\right| \leq \int_0^h \left|\frac{(t - h/2)^2}{2} - \frac{h^2}{8}\right| K_i dt = \frac{K_i}{2} \int_0^h |t(t - h)|\, dt = K_i\frac{h^3}{12}.$$

Finally we add up the error from each interval to get ($n$ intervals, $h = \frac{b-a}{n}$)

$$\left|\int_a^b f(x)dx - T_n\right| = n\frac{Kh^3}{12} = \frac{K(b - a)^3}{12n^2}.$$