Work

Work is defined as force times distance. When work is done on an object, it gains energy (lifting a weight, pushing someone on a swing, throwing a baseball, etc.). Common units are

$$\text{J (Joules)} = \text{N} \cdot \text{m (Netwon-meters)}, \ \text{ft-lbs (foot-pounds)}.$$

In some situations, you may have to calculate a volume, multiply by a density (mass per unit volume)to obtain a mass, and finally multiply by an acceleration to obtain a force: For instance the weight of a cubic meter of water near the earth's surface is

$$(1 \text{ m}^3)\left(1000 \ \frac{\text{kg}}{\text{m}^3}\right)\left(9.8 \ \frac{\text{m}}{\text{s}^2}\right) = 9800 \text{ N} \cdot \text{m}.$$

When the force involved is not constant, we can integrate to find the work done. Approximating the force as constant over short length intervals, summing the work over the short length intervals, and taking a limit as the size of the intervals goes to zero gives an integral

$$W \approx \sum_{i=1}^{n} F(x_i^*)\Delta x_i \xrightarrow{n\to\infty, \ \Delta x_i \to 0} \int_a^b F(x)dx.$$

There are various ways of breaking a problem up to calculate work, not necessarily following the above. It is usually left to you to implement a coordinate system, so your work may look different from someone else's, but the answer should come out the same.

Examples:

- An inverted conical tank of radius 4 m and height 10 m is filled to a height of 8 m with water. How much work is required to empty the tank? [Assume the water is siphoned from its surface.]

  We measure distance downwards from the top of the tank. The weight of water at a give depth x is

  $$(\pi r^2 dx)(1000)(9.8) \text{ N}.$$

  The distance this disc of water must be lifted is x meters. The radius of the disc at depth x can be found with using similar triangles

  $$\frac{4}{10} = \frac{r}{10-x}, \ r(x) = \frac{2}{5}(10-x) \text{ meters}.$$

  The water exists for depths $2 \le x \le 10$, hence the work required is

  $$\int_2^{10} \pi\left(\frac{2}{5}(10-x)\right)^2 x \ dx.$$

- A 100 ft cable of weight 2 lbs/ft hangs off the side of a building. What is the work required to lift all of the cable to the top of the building?

  We measure distance x from the top of the building downwards. The weight of a small piece of cable is $2dx$ lbs, and each small piece is lifted a distance of x ft, depending on how far down it is. Hence the required work is

  $$\int_0^{100} 2x \ dx.$$

1. [Section 6.6, exer. 13] A cable that weighs $2\frac{\text{lb}}{\text{ft}}$ is used to lift 800 lb of coal up a mine shaft 500 ft deep. Find the work done.

   The total work can be split into two parts

   $$W_{\text{total}} = W_{\text{coal}} + W_{\text{cable}}$$

   the work done to lift the coal (thought of as a point mass at the end of the cable) and the work to lift the cable itself. We have

   $$W_{\text{coal}} = (800 \text{ lbs})(500 \text{ ft}) = 400000 \text{ ft-lbs.}$$

   For the cable, we imagine lifting a small length of cable with weight $2dx$ lbs a distance of x ft (measured from the top of the mine shaft). Hence

   $$W_{\text{cable}} = \int_0^{500} 2x \; dx = 250000 \text{ ft-lbs.}$$

   The total work is then $W_{\text{total}} = 650000$ ft-lbs.

2. [Section 6.6, exer. 17] An aquarium 2 m long, 1 m wide, and 1 m deep is full of water. Find the work needed to pump half of the water out of the aquarium. (Use the fact that the density of water is $1000\frac{\text{kg}}{\text{m}^3}$ and that the acceleration due to gravity near the earth's surface is $9.8\frac{\text{m}}{\text{s}^2}$.)

   We imagine lifting all of the water at a given depth to the surface. The weight of water at a given depth x meters from the top of the aquarium is

   $$(2 \text{ m})(2 \text{ m})(dx \text{ m}) \left(1000 \; \frac{\text{kg}}{\text{m}^3}\right) \left(9.8\frac{\text{m}}{\text{s}^2}\right).$$

   There is water between $x = 1/2$ and $x = 1$. Hence

   $$W = 196000 \int_0^{1/2} x \; dx = 24500 \text{ J.}$$

3. The great pyramid of Giza is a pyramid with a square base 750ft $\times$ 750ft and height 480 ft. The density of the stone used in its construction is 200 lbs/ft^3. How much work was done constructing the pyramid? [Assume the pyramid is solid. How much stone must be lifted to a given height?] If one person can perform 2000 ft-lbs of work per hour, 10 hours per day, and the pyramid took 20 years to build, how many people were necessary for its construction?

   We think of building the pyramid by lifting thin square slabs of stone to a given height. The weight of the stone at height x meters from the ground is

$$200s^2 \, dx \text{ lbs}$$

   where $s = s(x)$ is the side length of the square obtained by slicing the pyramid perpendicular to its height. The side length can be computed using similar triangles (draw a triangle)

$$\frac{480}{375} = \frac{480 - x}{s/2}, \; s(x) = \frac{750}{480}(480 - x) \text{ ft.}$$

   Hence the work required is

$$W = 200 \int_0^{480} x \left(\frac{750}{480}(480 - x) \right)^2 \, dx = 2.16 \times 10^{12} \text{ ft-lbs.}$$

   Alternatively, the center of mass of a pyramid as at at one-fourth its height. So lifting the weight of the pyramid to its center of mass gives

$$\frac{480}{4} \cdot \frac{1}{3} \cdot 200 \cdot 750^2 \cdot 480 = 2.16 \times 10^{12} \text{ ft-lbs.}$$

   The number of people N needed to do this much work over a 20 year period (assuming 2000 fl-lbs of work per hour per person, 10 hours of work a day, and 365 workdays a year) satisfies

$$2.16 \times 10^{12} = N \cdot 2000 \cdot 10 \cdot 365 \cdot 20, \; N \approx 14794 \text{ people.}$$

4. How much energy is needed for an object of mass m to escape the earth's gravitational potential? In other words, what is the work done moving an object of mass m from the surface of the earth to infinity? The gravitational force between two bodies of masses M and m at a distance r from each other is $F(r) = GMm/r^2$. Converting this to kinetic energy, calculate the earth's escape veloicty. [If you want numbers, the mass of the earth is $M_e = 5.972 \times 10^{24}$ kg, the mean radius of the earth is $r_e = 6371$ km, and $G = 6.674 \times 10^{-11} \frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}$.]

   The energy needed is

$$\int_{r_e}^{\infty} \frac{GM_e m}{r^2} dr = \frac{GM_e m}{r_e} = m \cdot 6.25 \times 10^7 \text{ J.}$$

   converting this to kinetic energy gives an escape velocity of

$$m \cdot 6.25 \times 10^7 = \frac{1}{2} m v^2, \; v = 11185 \, \frac{\text{m}}{\text{s}},$$

   (that's pretty fast).