

Work

Work is defined as force times distance. When work is done on an object, it gains energy (lifting a weight, pushing someone on a swing, throwing a baseball, etc.). Common units are

$$\text{J (Joules)} = \text{N} \cdot \text{m (Newton-meters)}, \text{ ft-lbs (foot-pounds)}.$$

In some situations, you may have to calculate a volume, multiply by a density (mass per unit volume) to obtain a mass, and finally multiply by an acceleration to obtain a force: For instance the weight of a cubic meter of water near the earth's surface is

$$(1 \text{ m}^3) \left(1000 \frac{\text{kg}}{\text{m}^3} \right) \left(9.8 \frac{\text{m}}{\text{s}^2} \right) = 9800 \text{ N} \cdot \text{m}.$$

When the force involved is not constant, we can integrate to find the work done. Approximating the force as constant over short length intervals, summing the work over the short length intervals, and taking a limit as the size of the intervals goes to zero gives an integral

$$W \approx \sum_{i=1}^n F(x_i^*) \Delta x_i \xrightarrow{n \rightarrow \infty, \Delta x_i \rightarrow 0} \int_a^b F(x) dx.$$

There are various ways of breaking a problem up to calculate work, not necessarily following the above. It is usually left to you to implement a coordinate system, so your work may look different from someone else's, but the answer should come out the same.

Examples:

- An inverted conical tank of radius 4 m and height 10 m is filled to a height of 8 m with water. How much work is required to empty the tank? [Assume the water is siphoned from its surface.]

We measure distance downwards from the top of the tank. The weight of water at a give depth x is

$$(\pi r^2 dx)(1000)(9.8) \text{ N}.$$

The distance this disc of water must be lifted is x meters. The radius of the disc at depth x can be found with using similar triangles

$$\frac{4}{10} = \frac{r}{10 - x}, \quad r(x) = \frac{2}{5}(10 - x) \text{ meters}.$$

The water exists for depths $2 \leq x \leq 10$, hence the work required is

$$\int_2^{10} \pi \left(\frac{2}{5}(10 - x) \right)^2 x dx.$$

- A 100 ft cable of weight 2 lbs/ft hangs off the side of a building. What is the work required to lift all of the cable to the top of the building?

We measure distance x from the top of the building downwards. The weight of a small piece of cable is $2dx$ lbs, and each small piece is lifted a distance of x ft, depending on how far down it is. Hence the required work is

$$\int_0^{100} 2x dx.$$

1. [Section 6.6, exer. 13] A cable that weighs $2\frac{\text{lb}}{\text{ft}}$ is used to lift 800 lb of coal up a mine shaft 500 ft deep. Find the work done.

2. [Section 6.6, exer. 17] An aquarium 2 m long, 1 m wide, and 1 m deep is full of water. Find the work needed to pump half of the water out of the aquarium. (Use the fact that the density of water is $1000\frac{\text{kg}}{\text{m}^3}$ and that the acceleration due to gravity near the earth's surface is $9.8\frac{\text{m}}{\text{s}^2}$.)

3. The great pyramid of Giza is a pyramid with a square base $750\text{ft} \times 750\text{ft}$ and height 480 ft. The density of the stone used in its construction is 200 lbs/ft^3 . How much work was done constructing the pyramid? [Assume the pyramid is solid. How much stone must be lifted to a given height?] If one person can perform 2000 ft-lbs of work per hour, 10 hours per day, and the pyramid took 20 years to build, how many people were necessary for its construction?

4. How much energy is needed for an object of mass m to escape the earth's gravitational potential? In other words, what is the work done moving an object of mass m from the surface of the earth to infinity? The gravitational force between two bodies of masses M and m at a distance r from each other is $F(r) = GMm/r^2$. [If you want numbers, the mass of the earth is 5.972×10^{24} kg, the mean radius of the earth is 6371 km, and $G = 6.674 \times 10^{-11} \frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}$.]