Volumes of revolution

We discuss two ways of calculating the volume of a solid obtained by revolving a region in the plane about an axis.

- [Disks/Washers] Taking a region in the plane bounded by two curves $f(x) \geq g(x) \geq 0$ for $a \leq x \leq b$ and rotating it around the x-axis produces a solid of revolution. Slicing perpendular to the axis we obtain cross-sectional areas $A(x) = \pi(R^2 - r^2)$ that are "washers," defined by the radii $R = f(x)$ and $r = g(x)$. To obtain the volume, we integrate the cross-sectional areas:

$$\text{Volume} = \int_a^b A(x)dx = \int_a^b \pi(R^2 - r^2)dx = \int_a^b \pi(f(x)^2 - g(x)^2)dx.$$

- [Cylindrical Shells] Taking a region in the plane bounded by two curves $f(x) \geq g(x)$ for $0 \leq a \leq x \leq b$ and rotating it around the y-axis produces a solid of revolution. If we decompose the solid in the radial direction r ($r = x$ if rotating around the y-axis), we obtain radial cross-sections that are cylinders, of area $2\pi rh$ where the height of the cylinder is given by $f(x) - g(x)$ and the radius r is given by the distance to the axis, in this case the value of x. Adding up the volumes of approximating "cylindrical shells" and taking a limit give the volume as an integral of the areas of the radial cross-sections:

$$\text{Volume} = \int_a^b 2\pi rh\ dr = \int_a^b 2\pi x(f(x) - g(x))dx.$$

In both methods above, we could change how the regions are presented or what axis is the axis of revolution. Ingredients for computing such volumes include:

- Draw a picture and decide which method to use.
- Determine the axis of revolution and the variable of integration. For disks/washers, the variable of integration is along the axis of revolution (each point on the axis gives a cross-section). For cylindrical shells, the variable of integration is the radial direction, perpendicular to the axis of revolution (at each distance from the axis we have a cylindrical cross-section).
- Make sure the radii R, r (disk/washer) or radius and height r, h (cylindrical shells) are written as functions of the variable of integration. Moreover, for the radii in either method, make sure you are measuring distance to the axis (e.g. if the axis of revolution is not one of the coordinate axes, you might have to subtract a constant).

The following problems will all involve the region in the plane bounded by $y = x^2$ and $x = y^3$. Find the volume (or just set up the integral for the volume) of the solid obtained by revolving this region -

[To do the problems below, we will need to work with $y = y(x)$ and $x = x(y)$ (sometimes viewing y as a function of x and vice versa):

$$y = x^2 \Leftrightarrow \pm x = y^{1/2},\ x = y^3 \Leftrightarrow y = x^{1/3}.$$

We only need the positive square root $x = y^{1/2}$ because $x \geq 0$ in the region we consider.]

1. around the x-axis using disks/washers.

$$\int_0^1 \pi(R^2 - r^2)dx = \int_0^1 \pi[(x^{1/3})^2 - (x^2)^2]dx$$

2. around the x-axis using cylindrical shells.

$$\int_0^1 2\pi rh\ dy = \int_0^1 2\pi(y)(y^{1/2} - y^3)dy$$

3. around the line $y = 1$ using disks/washers.

$$\int_0^1 \pi(R^2 - r^2)dx = \int_0^1 \pi[(1 - x^2)^2 - (1 - x^{1/3})^2]dx$$

4. around the line $y = 1$ using cylindrical shells.

$$\int_0^1 2\pi rh\ dy = \int_0^1 2\pi(1 - y)(y^{1/2} - y^3)dy$$

5. around the y-axis using disks/washers.

$$\int_0^1 \pi(R^2 - r^2)dy = \int_0^1 \pi[(y^{1/2})^2 - (y^3)^2]dy$$

6. around the y-axis using cylindrical shells.

$$\int_0^1 2\pi rh\ dx = \int_0^1 2\pi(x)(x^{1/3} - x^2)dx$$

7. around the line $x = 1$ using disks/washers.

$$\int_0^1 \pi(R^2 - r^2)dy = \int_0^1 \pi[(1 - y^3)^2 - (1 - y^{1/2})^2]dy$$

8. around the line $x = 1$ using cylindrical shells.

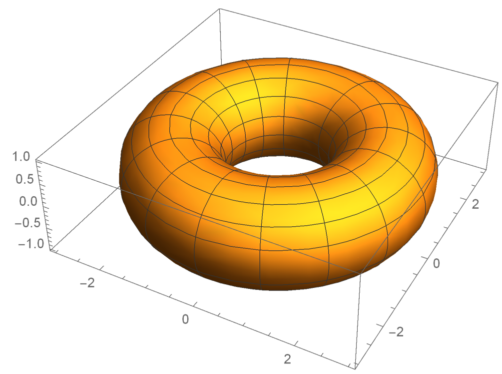
$$\int_0^1 2\pi rh\ dx = \int_0^1 2\pi(1 - x)(x^{1/3} - x^2)dx$$

9. [If you got through the above and have nothing to do...]

   (a) What is the volume of the region obtained by rotating the disk

$$(x-R)^2+y^2\le r^2,\ r\le R$$

   about the y-axis?



   Let's use shells, with height $2\sqrt{r^2-(x-R)^2}$ and radius x:

$$V=\int_{R-r}^{R+r} 2\pi x(2\sqrt{r^2-(x-R)^2})dx$$

   With the substitution $u=x-R$ we have

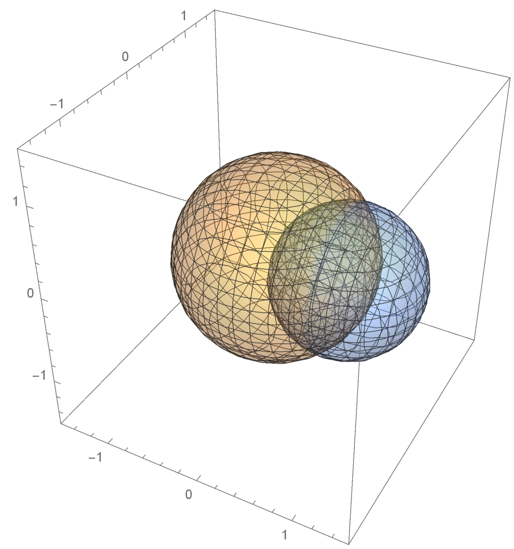
$$V=4\pi\int_{-r}^{r}(u+R)\sqrt{r^2-u^2}du=-2\pi\int_{-r}^{r}(-2u)\sqrt{r^2-u^2}du+4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}du$$
$$=0+4\pi Rr^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\ d\theta=4\pi Rr^2\int_{0}^{\pi/2}[1+\cos(2\theta)]d\theta=2\pi^2Rr^2.$$

   The "0" above comes from the fact that the integrand is odd and we're integrating over the interval $[-r,r]$, and symmetry was also used again with the even integrand $\cos^2\theta$. Or with washers,

$$V=\pi\int_{-r}^{r}\left[\left(R+\sqrt{r^2-y^2}\right)^2-\left(R-\sqrt{r^2-y^2}\right)^2\right]dy=8R\pi\int_{0}^{r}\sqrt{r^2-y^2}dy$$
$$=2\pi^2r^2R,$$

   recognizing the last integral as the area of a quarter-disk with radius r, or you can integrate using an inverse trigonometric substitution ($y=r\sin\theta$).

(b) What is the volume of the intersection of two spheres with radii R and r if the distance between their centers is d?



The x-coordinate of the intersection of the circles (centering one sphere at the origin and the other at $(x, y, z) = (d, 0, 0)$)

$$x^2 + y^2 = R^2, \ (x - d)^2 + y^2 = r^2,$$

is

$$x_0 = \frac{R^2 - r^2 + d^2}{2d}.$$

Cross-sections perpendicular to the x-axis are disks of radius

$$y = \sqrt{r^2 - (x - d)^2}, \ d - r \le x \le x_0, \ y = \sqrt{R^2 - x^2}, \ x_0 \le x \le d + r.$$

The volume is

$$\begin{aligned} V &= \int_{d-r}^{x_0} \pi(\sqrt{r^2 - (x-d)^2})^2 dx + \int_{x_0}^{d+r} \pi(\sqrt{R^2 - x^2})^2 dx \\ &= \pi \int_{d-r}^{x_0} (r^2 - (x-d)^2) dx + \pi \int_{x_0}^{d+r} (R^2 - x^2) dx \\ &= \pi r^2 (x_0 - d + r) - \frac{\pi}{3}((x_0 - d)^3 + r^3) + \pi R^2 (d + r - x_0) - \frac{\pi}{3}((d + r)^3 - (x_0)^3) \\ &= \ldots = \frac{\pi}{12d}(R + r - d)^2 (d^2 + 2d(R + r) - 3(R - r)^2). \end{aligned}$$

For example, if $R = r = d$ then $V = \frac{5\pi}{12} R^3$.