Volume as the integral of cross-sectional area

If a solid body B is extended along some axis, say $a \le x \le b$, and for each x the area of the cross-section perpendicular to the axis is $A(x)$, then we can approximate the volume of B with a Riemann sum

$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x_i$$

where we partition the interval $[a, b]$ ito n pieces and choose sample points x_i^* in each interval

$$x_0 = a < x_1 < x_2 < \ldots < x_n = b,\ x_{i-1} \le x_i^* \le x_i,\ \Delta x_i = x_i - x_{i-1}.$$

Assuming A is continuous as a function of x, the limit of these approximations (as $n \to \infty$ and $\max\{\Delta x_i : 1 \le i \le n\} \to 0$) becomes an integral, which we take to be the volume of the body B

$$V := \int_a^b A(x)dx.$$

To summarize, we obtain the volume of B by integrating its cross-sectional areas $A(x)$ with respect to x along some axis.

Examples:

- We can find the volume of a sphere $4\pi R^3/3$ by centering it at the origin and cutting perpendicular to the x-axis to get cross-sections that are disks of area $A(x) = \pi r^2$, with $r = \sqrt{R^2 - x^2}$. We get

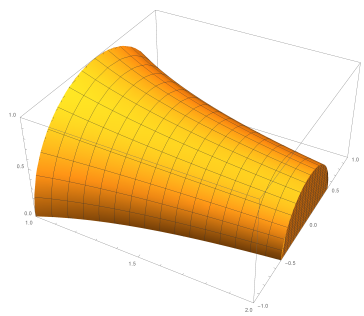
$$V = \int_{-R}^{R} \pi(R^2 - x^2)dx = 2\pi \int_0^R (R^2 - x^2)dx = 2\pi(R^2x - x^3/3)\Big|_0^R = 4\pi R^3/3.$$

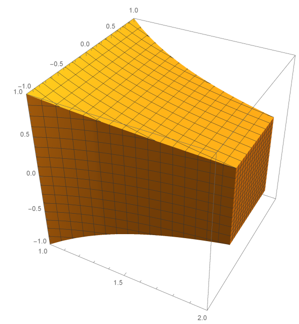
- We can find the volume of a right cone of height h (with any base area A) by cutting perpendicular to its height. The cross-sectional area will be $A(x) = \left(\frac{x}{h}\right)^2 A$ (if we put the "point" of the cone at $x = 0$). Hence the volume is

$$\int_0^h \left(\frac{x}{h}\right)^2 Adx = A\frac{x^3}{3h^2}\Big|_0^h = \frac{1}{3}Ah,$$

i.e. the volume is "one-third base times height" as expected.

- What are the volumes of the solids pictured below? [The cross-sections are half-disks and squares, the radii and half the side-lengths are given by $f(x) = 1/x$ with $1 \leq x \leq 2$]. What if we let $1 \leq x < \infty$?





When the cross-sections perpendicular to the x-axis are squares of sidelength s, the cross-sectional area is

$$A(x) = s^2 = (2/x)^2$$

and the volume is given by the integral

$$\int_1^2 A(x)dx = \int_1^2 s^2 dx = \int_1^2 (2/x)^2 dx = 2.$$

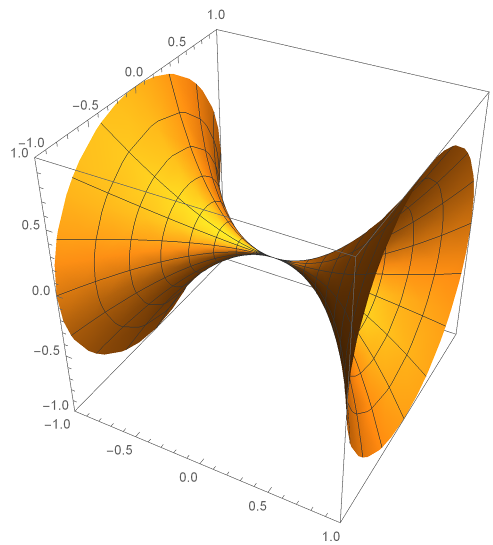
When the cross-sections perpendicular to the x-axis are half-disks or radius r, the cross-sectional area is

$$A(x) = \frac{\pi}{2} r^2 = \frac{\pi}{2}(1/x)^2$$

and the volume is

$$\int_1^2 A(x)dx = \int_1^2 \frac{\pi}{2} r^2 dx = \int_1^2 \frac{\pi}{2}(1/x)^2 dx = \frac{\pi}{4}.$$

- What is the volume of the solid obtained by rotating the graph of $y = x^2$, $-1 \le x \le 1$, around the x-axis? [The cross-sections will be disks of radius y.]



Cross-sections perpendicular to the x-axis are disks with area πr^2, where the radius r is $y = x^2$. Every value $-1 \le x \le 1$ determines a disk. Integrating the cross-sectional areas along the x-axis gives the volume

$$\int_{-1}^{1} \pi r^2 dx = \int_{-1}^{1} \pi (x^2)^2 dx = \frac{2\pi}{5}.$$