

Volume as the integral of cross-sectional area

If a solid body B is extended along some axis, say $a \leq x \leq b$, and for each x the area of the cross-section perpendicular to the axis is $A(x)$, then we can approximate the volume of B with a Riemann sum

$$V \approx \sum_{i=1}^n A(x_i^*) \Delta x_i$$

where we partition the interval $[a, b]$ into n pieces and choose sample points x_i^* in each interval

$$x_0 = a < x_1 < x_2 < \dots < x_n = b, \quad x_{i-1} \leq x_i^* \leq x_i, \quad \Delta x_i = x_i - x_{i-1}.$$

Assuming A is continuous as a function of x , the limit of these approximations (as $n \rightarrow \infty$ and $\max\{\Delta x_i : 1 \leq i \leq n\} \rightarrow 0$) becomes an integral, which we take to be the volume of the body B

$$V := \int_a^b A(x) dx.$$

To summarize, we obtain the volume of B by integrating its cross-sectional areas $A(x)$ with respect to x along some axis.

Examples:

- We can find the volume of a sphere $4\pi R^3/3$ by centering it at the origin and cutting perpendicular to the x -axis to get cross-sections that are disks of area $A(x) = \pi r^2$, with $r = \sqrt{R^2 - x^2}$. We get

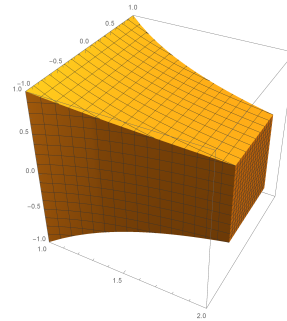
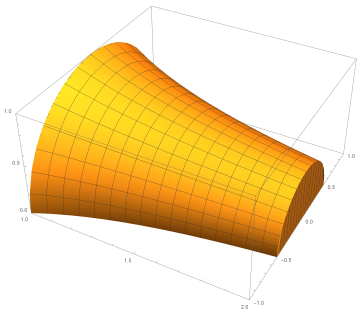
$$V = \int_{-R}^R \pi(R^2 - x^2) dx = 2\pi \int_0^R (R^2 - x^2) dx = 2\pi(R^2 x - x^3/3) \Big|_0^R = 4\pi R^3/3.$$

- We can find the volume of a right cone of height h (with any base area A) by cutting perpendicular to its height. The cross-sectional area will be $A(x) = \left(\frac{x}{h}\right)^2 A$ (if we put the “point” of the cone at $x = 0$). Hence the volume is

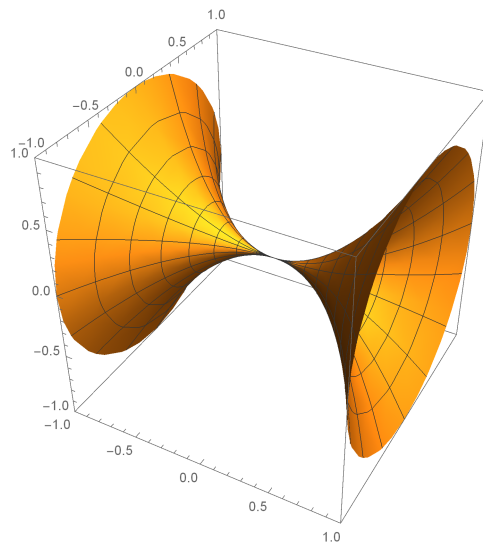
$$\int_0^h \left(\frac{x}{h}\right)^2 A dx = A \frac{x^3}{3h^2} \Big|_0^h = \frac{1}{3} Ah,$$

i.e. the volume is “one-third base times height” as expected.

- What are the volumes of the solids pictured below? [The cross-sections are half-disks and squares, the radii and half the side-lengths are given by $f(x) = 1/x$ with $1 \leq x \leq 2$]. What if we let $1 \leq x < \infty$?



- What is the volume of the solid obtained by rotating the graph of $y = x^2$, $-1 \leq x \leq 1$, around the x -axis? [The cross-sections will be disks of radius y .]



- What is the volume of a regular tetrahedron with side length a ? (Answer: $\frac{a^3}{6\sqrt{2}}$.)