Trigonometric substitution

Expressions such as

$$\sqrt{a^2 - x^2},\ \sqrt{a^2 + x^2},\ \sqrt{x^2 - a^2},\ a > 0 \text{ constant},$$

may remind you of writing one side of a right triangle in terms of the other

$$a^2 + b^2 = c^2 \Longrightarrow c = \sqrt{a^2 + b^2},\ a = \sqrt{c^2 - b^2}.$$

We can eliminate the $\sqrt{\ }$ with the Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1,\ \tan^2\theta + 1 = \sec^2\theta,\ (1 + \cot^2\theta = \csc^2\theta)$$

by parameterizing x with trigonometric function

$$x = a\sin\theta,\ x = a\tan\theta,\ x = a\sec\theta.$$

With these substitutions, we have (for appropriate ranges of θ where, e.g., $\sqrt{\sin^2\theta} = |\sin\theta| = \sin\theta$)

$$\begin{aligned}
\sqrt{a^2 - x^2} &= \sqrt{a^2 - (a\sin\theta)^2} = a\sqrt{1 - \sin^2\theta} = a\sqrt{\cos^2\theta} = a\cos\theta \\
\sqrt{a^2 + x^2} &= \sqrt{a^2 + (a\tan\theta)^2} = a\sqrt{1 + \tan^2\theta} = a\sqrt{\sec^2\theta} = a\sec\theta \\
\sqrt{x^2 - a^2} &= \sqrt{(a\sec\theta)^2 - a^2} = a\sqrt{\sec^2\theta - 1} = a\sqrt{\tan^2\theta} = a\tan\theta.
\end{aligned}$$

We can use the above to integrate some new functions. The answers you get can usually be rewritten in several ways, for instance

$$\arccos(x) = \arcsin(\sqrt{1 - x^2}) = \arctan(\sqrt{1/x^2 - 1}), \text{etc.}$$

We could also use the "hyperbolic" trigonometic functions (described in the next section), but they are not strictly part of the cirriculum.

1. $\displaystyle\int \sqrt{1-x^2}dx$

With $x = \sin\theta$, $dx = \cos\theta\ d\theta$, we have

$$\int \sqrt{1-x^2}dx = \int \sqrt{1-\sin^2\theta}\cos\theta d\theta = \int \cos^2\theta d\theta = \frac{1}{2}\int (1+\cos(2\theta))d\theta$$
$$= \frac{\theta}{2} + \frac{\sin(2\theta)}{4} = \frac{1}{2}(\theta + \sin\theta\cos\theta).$$

Now, we need to get back the variable $x = \sin\theta$, $\theta = \arcsin x$. The right triangle with angle θ, hypothenuse 1, opposite side x and adjacent side $\sqrt{1-x^2}$ (draw a picture) satisfies $x = \sin\theta$ and tells us that $\cos\theta = \sqrt{1-x^2}$. Hence

$$\frac{1}{2}(\theta + \sin\theta\cos\theta) = \frac{1}{2}(\arcsin x + x\sqrt{1-x^2}).$$

2. $\displaystyle\int \frac{dx}{\sqrt{x^2-4}}$

With $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta d\theta$, we have

$$\int \frac{dx}{\sqrt{x^2-4}} = \int \frac{2\sec\theta\tan\theta d\theta}{\sqrt{4\sec^2\theta - 4}} = \int \frac{2\sec\theta\tan\theta d\theta}{2\sqrt{\sec^2\theta - 1}} = \int \frac{\sec\theta\tan\theta d\theta}{\sqrt{\tan^2\theta}}$$
$$= \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta|.$$

We need to switch back to the variable $x = 2\sec\theta$, with $\theta = \operatorname{arcsec}(x/2)$. The right triangle with angle θ, hypothenuse x, adjacent side 2, and opposite side $\sqrt{x^2-4}$ (draw a picture) satisfies $2\sec\theta = x$ and shows that $\tan\theta = \frac{1}{2}\sqrt{x^2-4}$. Hence

$$\ln|\sec\theta + \tan\theta| = \ln\left|\frac{x}{2} + \frac{1}{2}\sqrt{x^2-4}\right|.$$

3. Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

The top half of the ellipse is given by the graph of $y = b\sqrt{1-(x/a)^2}$. The area of one quarter of the ellipse is given by

$$\int_0^a b\sqrt{1-(x/a)^2}dx.$$

With $x = a\sin\theta$, $dx = a\cos\theta d\theta$, we have

$$\int_0^a b\sqrt{1-(x/a)^2}dx = \int_0^{\pi/2} b\sqrt{1-\sin^2\theta}\ a\cos\theta d\theta = ab\int_0^{\pi/2} \cos^2\theta\ d\theta$$
$$= \frac{ab}{2}\int_0^{\pi/2}(1+\cos(2\theta))d\theta = \frac{ab}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right)\Big|_0^{\pi/2} = \frac{\pi ab}{4}.$$

Hence the total area of the ellipse is πab. When $a = b = r$ (i.e. the ellipse is a circle of radius r), we get πr^2 as expected.

4. $\displaystyle\int \frac{dx}{x\sqrt{x^2-9}}$

With $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta d\theta$, we have

$$\int \frac{dx}{x\sqrt{x^2-9}} = \int \frac{3\sec\theta\tan\theta d\theta}{3\sec\theta\sqrt{9\sec^2\theta-9}} = \frac{1}{3}\int \frac{\sec\theta\tan\theta d\theta}{\sec\theta\sqrt{\sec^2\theta-1}} = \frac{1}{3}\int \frac{\sec\theta\tan\theta d\theta}{\sec\theta\sqrt{tan^2\theta}}$$
$$= \frac{1}{3}\int d\theta = \frac{\theta}{3}.$$

Switching to x we get

$$\frac{\theta}{3} = \frac{1}{3}\operatorname{arcsec}(x/3).$$

5. $\displaystyle\int \frac{dx}{\sqrt{x^2+4x+20}}$

We complete the square and substitute to get

$$\int \frac{dx}{\sqrt{x^2+4x+20}} = \int \frac{dx}{\sqrt{(x+2)^2+16}} = \int \frac{du}{\sqrt{u^2+4^2}}.$$

With $u = 4\tan\theta$, $du = 4\sec^2\theta d\theta$, we have

$$\int \frac{du}{\sqrt{u^2+4^2}} = \int \frac{4\sec^2\theta d\theta}{\sqrt{16\tan^2\theta+16}} = \int \frac{\sec^2\theta d\theta}{\sqrt{\tan^2\theta+1}} = \int \frac{\sec^2\theta d\theta}{\sqrt{\sec^2\theta}}$$
$$= \int \sec\theta d\theta = \ln|\sec\theta+\tan\theta|.$$

Switching back to $u = 4\tan\theta$ (triangles blahblahblah) and then to $x = u - 2$ we get

$$\ln|\sec\theta+\tan\theta| = \ln|\sqrt{1+(u/4)^2}+u/4| = \ln|\sqrt{1+((x+2)/4)^2}+(x+2)/4|.$$

Extra-cirricular: hyperbolic trigonometric functions

The parameterization $x = \cos t$, $y = \sin t$ nicely discribes the unit circle, $x^2 + y^2 = 1$; as t increases, $(x(t), y(t))$ moves counter-clockwise around the circle at a constant speed.

There is a similar parameterization of the unit hyperbola $x^2 - y^2 = 1$ by the "hyperbolic" (as opposed to "circular") trigonometric functions, definied as

$$\cosh t = \frac{e^t + e^{-t}}{2}, \ \sinh t = \frac{e^t - e^{-t}}{2}.$$

One can verify that

$$\cosh^2 t - \sinh^2 t = 1, \ \frac{d}{dt} \cosh t = \sinh t, \ \frac{d}{dt} \sinh t = \cosh t,$$

(see page 227 of your text). So it seems natural to use these to integrate functions involving $\sqrt{x^2 - a^2}$ and $\sqrt{a^2 + x^2}$ (or even $\sqrt{a^2 - x^2}$ if we use $\tanh t$ and $\operatorname{sech} t$).

The hyperbolic trigonometric functions are closely related to the circular trigonometric functions, in fact we have

$$\sin t = \frac{e^{it} - e^{-it}}{2i} = -i \sinh(it), \ \cos t = \frac{e^{it} + e^{-it}}{2} = \cosh(it), \ i = \sqrt{-1},$$

which we won't discuss, but could prove after we discuss power series later in the course. Another similarity comes from the fact that, $\sin t$, $\cos t$ are the solutions to the differential equations $y'' + y = 0$ with initial conditions $y(0) = 0$, $y'(0) = 1$ and $y(0) = 1$, $y'(0) = 0$ respectively, where as $\sinh t$ and $\cosh t$ are solutions of $y'' - y = 0$ with the same initial conditions. We will discuss some differential equation later in the course.

Extra-Extra-cirricular: $e^{ix} = \cos x + i \sin x$

We will see later in the course that the exponential function e^x has the power series expansion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots, \quad \text{where } n! = n(n-1)(n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1,$$

whereas $\sin x$ and $\cos x$ have series expansions

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots, \ \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots$$

Replacing x by ix in e^x (where $i = \sqrt{-1}$) gives

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n \text{ even}} \frac{(ix)^n}{n!} + \sum_{n \text{ odd}} \frac{(ix)^n}{n!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \\ &= \cos x + i \sin x \end{aligned}$$

where we write $n = 2k$ for n even and $n = 2k+1$ for n odd, so that $i^{2k} = (-1)^k$, $i^{2k+1} = i(-1)^k$.