Integrating trigonometric functions

Here are the most basic trigonometric identities

$$\sin^2 x + \cos^2 x = 1 \text{ (Pythagorean identity)}$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

From these follow

$$\tan^2 x + 1 = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$
$$\sin(2x) = 2 \sin x \cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x.$$

and

$$\cos^2 x = \frac{1+\cos(2x)}{2}, \ \sin^2 x = \frac{1-\cos(2x)}{2}.$$

We can use these along with our other integration techniques to integrate new functions. [Appendix C of the text contains a review of trigonometry.]

1. Integrate $\int \sin x \cos x \, dx$ two ways, with a substitution and with a trig identity. Here are

   three ways: Let $u = \sin x$, $du = \cos x \, dx$ to get

   $$\int \sin x \cos x \, dx = \int u \, du = u^2/2 + C = \sin^2 x/2 + C.$$

   Let $u = \cos x$, $du = -\sin x \, dx$ to get

   $$\int \sin x \cos x \, dx = -\int u \, du = -u^2/2 + C = -\cos^2/2 + C.$$

   Use $\sin(2x) = 2 \sin x \cos x$ to get

   $$\int \sin x \cos x \, dx = \frac{1}{2} \int \sin(2x) dx = -\cos(2x)/4 + C.$$

   Exercise: Use trig identities to show that these answers are all equivalent (i.e. differ by a constant).

2. $\int \cos^5 x \sin^4 x \, dx$

   Let $u = \sin x$, $du = \cos x \, dx$ and use $\cos^2 x - 1 = \sin^2 x$ to get

   $$\int \cos^5 x \sin^4 x \, dx = \int (\cos^2)^2 x \sin^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 \sin^4 x \cos x \, dx$$
   $$= \int (1-u^2)^2 u^4 du = \int (u^4 - 2u^6 + u^8) du = \sin^5 x/5 - 2\sin^7 x/7 + \sin^9 x/9 + C.$$

3. $\int \sin^3 x \; dx$
We have (with $u = \cos x$, $du = -\sin x \; dx$)

$$\int \sin^3 x \; dx = \int (1-\cos^2 x) \sin x \; dx = -\int (1-u^2) du = u - u^3/3 + C = \cos x - \cos^3 x/3 + C.$$

4. $\int \sec^4 x \tan^3 x \; dx$
With $u = \sec x$, $du = \sec x \tan x \; dx$, we have

$$\int \sec^4 x \tan^3 x \; dx = \int \sec^3 x \tan^2 x \sec x \tan x \; dx = \int \sec^3 x (\sec^2 x - 1) \sec x \tan x \; dx$$
$$= \int u^3 (u^2 - 1) du = u^6/6 - u^4/4 + C = \sec^6 x/6 - \sec^4 x/4 + C.$$

5. $\int \sec^2 x \tan^3 x \; dx$ (problem differs from the handout in class)
With $u = \tan x$, $du = \sec^2 x \; dx$, we get

$$\int \sec^2 x \tan^3 x \; dx = \int u^3 \; du = u^4/4 + C = \tan^4 x/4 + C.$$

6. $\int \cot^3 x \csc^3 x \; dx$ (problem differs from the handout in class)
With $u = \csc x$, $du = -\csc x \cot x \; dx$, we get

$$\int \cot^3 x \csc^3 x \; dx = \int \cot^2 x \csc^2 x \csc x \cot x \; dx = \int (\csc^2 x - 1) \csc^2 x \csc x \cot x \; dx$$
$$= -\int (u^2 - 1) u^2 du = -u^5/5 + u^3/3 + C = -\csc^5 x/5 + \csc^3 x/3 + C.$$