Taylor series

The question we'd like to address is:

Given a function $f(x)$ and a point a, can we write $f(x)$ as a power series centered at a,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n?$$

Assume this is the case. Repeatedly differentiating f gives us the following information:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} c_n (x-a)^n \\
f(a) &= \sum_{n=0}^{\infty} c_n (a-a)^n = c_0 (a-a)^0 + c_1 (a-a)^1 + \ldots = c_0 \\
f'(x) &= \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \\
f'(a) &= \sum_{n=1}^{\infty} n c_n (a-a)^{n-1} = 1 \cdot c_1 (a-a)^0 + 2 \cdot c_2 (a-a)^1 + \ldots = 1 \cdot c_1 \\
f''(x) &= \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2} \\
f''(a) &= \sum_{n=2}^{\infty} n(n-1) c_n (a-a)^{n-2} = 2 \cdot 1 \cdot c_2 (a-a)^0 + 3 \cdot 2 \cdot c_3 (a-a)^1 + \ldots = 2 \cdot 1 \cdot c_2 \\
&\ldots \\
f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)(n-2)\ldots(n-k+1) c_n (x-a)^{n-k} \\
f^{(k)}(a) &= \sum_{n=k}^{\infty} n(n-1)(n-2)\ldots(n-k+1) c_n (a-a)^{n-k} = k! c_k.
\end{aligned}$$

In other words **the coefficients of the series are determined by the derivatives of f at a** by the formula

$$c_k = \frac{f^{(k)}(a)}{k!}.$$

You should remember this expression from our discussion of Taylor polynomials. If a function has a power series representation near a, then the power series must be

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = \lim_{n \to \infty} T_n(x),$$

the series whose partial sums are the Taylor polynomials associated to f near a. This series is known as the Taylor series of f centered at a.

So *if* f can be represented as a power series near a, the power series *must* be the Taylor series of f at a. However, not every infinitely differentiable function can be written as a power series, and there is work to be done to show this is the case for the examples we'll work with.

Find the Taylor series centered at zero for the following functions and their interval of convergence:

1. e^x

   We have $f^{(n)}(x) = e^x$, $f^{(n)}(0) = e^0 = 1$ for all n. Hence the Taylor series is
$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$
   This power series has radius of convergence $R = \infty$.

2. $\sin x$

   We have
$$f^{(n)}(x) = \begin{cases} \sin x & n = 4k \\ \cos x & n = 4k+1 \\ -\sin x & n = 4k+2 \\ -\cos x & n = 4k+3 \end{cases},$$
   i.e. the derivative is periodic with period 4. Evaluating at $x = 0$ gives
$$f^{(n)}(0) = \begin{cases} 0 & n = 4k \\ 1 & n = 4k+1 \\ 0 & 4k+2 \\ -1 & 4k+3 \end{cases}.$$
   Hence the Taylor series is
$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$
   where we indexed so as not to include the zero coefficients of x^{2n}. This power series has radius of convergence $R = \infty$.

3. $\cos x$

   We have
$$f^{(n)}(x) = \begin{cases} \cos x & n = 4k \\ -\sin x & n = 4k+1 \\ -\cos x & n = 4k+2 \\ \sin x & n = 4k+3 \end{cases},$$
   so that
$$f^{(n)}(0) = \begin{cases} 1 & n = 4k \\ 0 & n = 4k+1 \\ -1 & n = 4k+2 \\ 0 & n = 4k+3 \end{cases}$$
   Hence the Taylor series is
$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$
   where we indexed so as not to include the zero coefficients of x^{2n+1}. This power series has radius of convergence $R = \infty$.

4. $\ln(1+x)$ (from scratch or start with the geometric series)

For $n \geq 1$ we have $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^{n+1}}$, $f^{(n)}(0) = (-1)^{n-1}(n-1)!$. Hence the Taylor series is

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

This power series has radius of convergence $R = 1$ and interval of convergence $(-1, 1]$.

Or, starting from the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ we get

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n, \; \ln(1+x) + C = \int \left(\sum_{n=0}^{\infty} (-1)^n (x)^n \right) dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

with $C = 0$ (evaluating at $x = 0$). The radius of convergence of the resulting series is $R = 1$ since that was the radius of convergence for the geometric series.

5. $\arctan x$ (start with the geometric series)

Starting from the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ we get

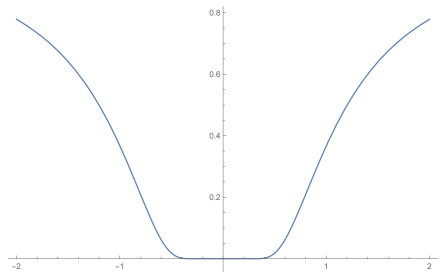
$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n, \; \arctan x + C = \int \left(\sum_{n=0}^{\infty} (-1)^n (x)^{2n} \right) dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

with $C = 0$ (evaluating at $x = 0$). The radius of convergence of the resulting series is $R = 1$ since that was the radius of convergence for the geometric series. The resulting series has interval of convergence $[-1, 1]$.

Extracirricular: a non-analytic C^∞ function

Here is an example of an infinitely differentiable function that cannot be expressed as a power series near zero:

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} .$$



As you can see, this function is very "flat" at $x = 0$; in fact we have $f^{(n)}(0) = 0$, every derivative of f at zero is equal to zero. So the Taylor series of f at $x = 0$ is $\sum_{n=0}^{\infty} 0 \cdot x^n = 0$, the zero function. In particular, f is not equal to its Taylor series at zero, so f cannot be written as a power series centered at zero.