Taylor[1] polynomials

The best constant approximation to a function $f(x)$ near $x = a$ is the constant function

$$T_0(x) = f(a)$$

The best linear approximation to f near $x = a$ is the tangent line approximation

$$T_1(x) = f(a) + f'(a)(x - a)$$

which uses the value of f at a and the value of the first derivative of f at a.

The nth degree Taylor polynomial of f centered at $x = a$ is the best nth degree polynomial approximation to f near $x = a$

$$\begin{aligned} T_n(x) &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k, \end{aligned}$$

assuming f is n times differentiable at $x = a$. [The notation $f^{(n)}$ means the nth derivative of f, and $f^{(0)}$ means the zeroth derivative, i.e. the function itself.] The Taylor polynomial T_n is determined by the numbers

$$f(a),\ f'(a),\ \ldots,\ f^{(n)}(a).$$

The Taylor polynomial T_n is the unique nth degree polynomial whose derivatives of order $\leq n$ at $x = a$ agree with the derivatives of f at $x = a$:

$$T_n(a) = f(a),\ T_n'(a) = f'(a),\ T''(a) = f''(a), \ldots,\ T_n^{(n)}(a) = f^{(n)}(a).$$

By "best" approximation, we mean

$$\lim_{x \to a} \frac{f(x) - T_n(x)}{(x-a)^n} = 0.$$

In fact we have

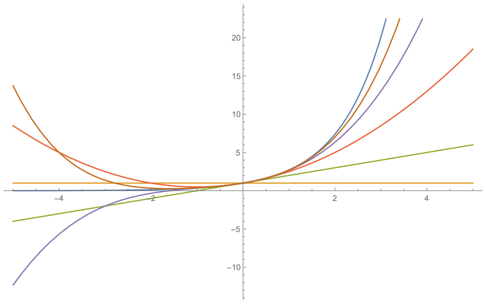
$$f(x) - T_n(x) = \frac{1}{n!}\int_a^x (x-t)^{n+1} f^{(n+1)}(t)dt = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

for some z between x and a. [The first of these follows from integration by parts, and the second is a version of the mean value theorem. Proofs can be found here.]

Find the fifth degree Taylor polynomials centered at $a = 0$ for the following functions. Try to establish a pattern for the coefficients and express $T_n(x)$ in summation notation.

[1]Brook Taylor, (1685-1731)

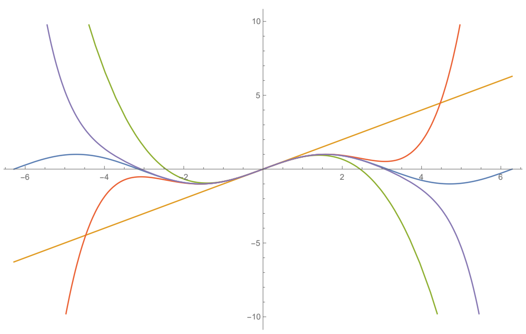
$f(x) = e^x$



n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$
0	e^x	1	1
1	e^x	1	1
2	e^x	1	1/2
3	e^x	1	1/6
4	e^x	1	1/24
5	e^x	1	1/120

$$T_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

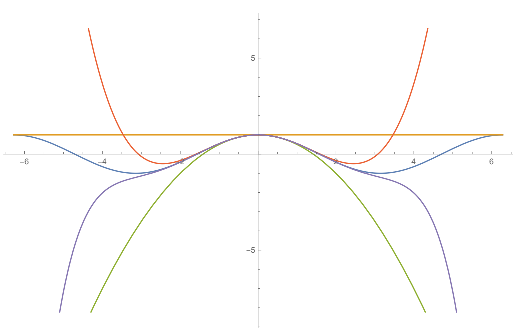
$f(x) = \sin x$



n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$
0	$\sin x$	0	0
1	$\cos x$	1	1
2	$-\sin x$	0	0
3	$-\cos x$	-1	-1/6
4	$\sin x$	0	0
5	$\cos x$	1	1/120

$$T_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$$

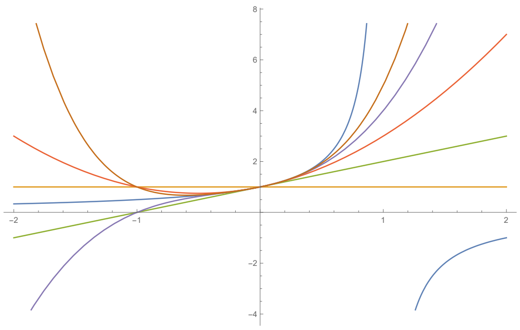
$f(x) = \cos x$



n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$
0	$\cos x$	1	1
1	$-\sin x$	0	0
2	$-\cos x$	-1	-1/2
3	$\sin x$	0	0
4	$\cos x$	1	1/24
5	$-\sin x$	0	0

$$T_5(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

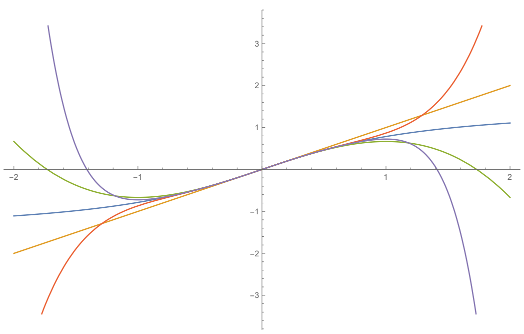
$f(x) = \frac{1}{1-x}$



n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$
0	$\frac{1}{1-x}$	1	1
1	$\frac{1}{(1-x)^2}$	1	1
2	$\frac{2}{(1-x)^3}$	2	1
3	$\frac{3\cdot 2\cdot 1}{(1-x)^4}$	6	1
4	$\frac{4!}{(1-x)^5}$	$4!$	1
5	$\frac{5!}{(1-x)^6}$	$5!$	1

$$T_5(x) = 1 + x + x^2 + x^3 + x^4 + x^5$$

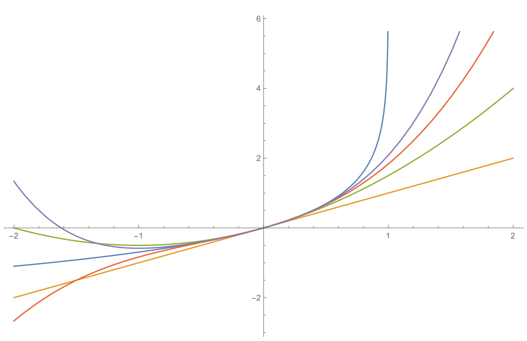
$f(x) = \arctan x$



n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$
0	$\arctan x$	0	0
1	$\frac{1}{1+x^2}$	1	1
2	$\frac{-2x}{(1+x^2)^2}$	0	0
3	$\frac{6x^2-2}{(1+x^2)^3}$	-2	-1/3
4	$\frac{24x(1-x^2)}{(1+x^2)^4}$	0	0
5	$\frac{24(5x^4-10x^2+1)}{(1+x^2)^5}$	24	1/5

$$T_5(x) = x - \frac{x^3}{3} + \frac{x^5}{5}$$

$f(x) = -\ln(1-x)$



n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$
0	$=\ln(1-x)$	0	0
1	$\frac{1}{1-x}$	1	1
2	$\frac{1}{(1-x)^2}$	1	1/2
3	$\frac{2}{(1-x)^3}$	2	1/3
4	$\frac{6}{(1-x)^4}$	6	1/4
5	$\frac{4!}{(1-x)^5}$	$4!$	1/5

$$T_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}$$