Series

Given a sequence $(a_n)_{n=0^\infty}$ we can try to add up all of the numbers in the list (in order). That is, we add up the first N terms of the sequence and take a limit as $N \to \infty$:

$$\sum_{n=0}^{\infty} a_n := \lim_{N\to\infty} \sum_{n=0}^{N} a_n = \lim_{N\to\infty} (a_0 + a_1 + \ldots + a_N).$$

If we can write down a fomula for the Nth partial sum $S_N = \sum_{n=0}^{N} a_n$ and take a limit as $N \to \infty$, then we obtain the value of the infinite series. (Unfortunately this is not often the case and we will spend the next week or two discussing methods for telling whether or not the limit even exists at all.) The simplest necessary condition for the limit to exist is that the terms must be getting smaller and smaller, $\lim_{n\to\infty} a_n = 0$.

$$\text{If } \lim_{n\to\infty} a_n \neq 0, \text{ then } \sum_{n=0}^{\infty} a_n \text{ does not converge. (Divergence Test)}$$

Examples:

- A geometric series is one whose adjacent terms have a fixed common ration r, i.e. $a_{n+1}/a_n = r$. So we want to find the partial sums

  $$S_N = a + ar + ar^2 + \ldots + ar^N.$$

  Multiplying by r gives

  $$rS_N = ar + ar^2 + \ldots + ar^N + ar^{N+1} = S_N - a + ar^{N+1}.$$

  Solving for S_N gives (assuming $r \neq 1$)

  $$S_N = a\frac{1 - r^{N+1}}{1 - r}.$$

  Taking limits as $N \to \infty$ gives

  $$a\sum_{n=0}^{\infty} r^n = \lim_{N\to\infty} a\frac{1 - r^{N+1}}{1 - r} = \frac{a}{1 - r} \text{ if } |r| < 1.$$

- A "telescoping" series is one in which there is a lot of cancellation, making the partial sums easier to find. For instance

  $$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \lim_{N\to\infty} \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots \left(\frac{1}{N-1} - \frac{1}{N}\right) + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= \lim_{N\to\infty} 1 - \frac{1}{N} = 1.\end{aligned}$$

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3 \cdot 2^{2n+1}}{5 \cdot 3^{2n-1}}$ (What is the first term? What is the common ratio?)

2. Write the repeating decimal $0.\overline{123}$ as a fraction.

3. $\displaystyle\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$ (This series is telescoping.)