Sequences

A **sequence** is a list of (real) numbers,

$$(a_n)_{n=1}^{\infty} = (a_1, a_2, a_3, \ldots),$$

usually indexed by a subset of the natural numbers $0, 1, 2, 3, \ldots$. A sequence **converges** to a limit L if for any error $\epsilon > 0$ there is an index N (depending on ϵ) so that

$$|a_n - L| < \epsilon \text{ whenever } n \geq N.$$

In other words the terms of the sequence get arbitrarily close to L for n sufficiently large.

A sequence is **bounded** if there is a bound M such that $|a_n| \leq M$ for any n. A sequence is **monotone** increasing (decreasing) if $a_n \leq a_{n+1}$ ($a_n \geq a_{n+1}$). Arguably the most important property of the real numbers is the following:

Any bounded monotone sequence converges.

The usual limit laws apply, e.g.

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n, \ \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$$

if the limits all exist and f is continuous at $\lim_{n\to\infty} a_n$.

Here are some random examples.

- The sequence

  $$0, 1, 0, -1, 0, 1, 0, -1, \ldots$$

  is periodic and does not converge. There are various ways for writing a formula for this sequence, such as

  $$\left\{\sin\left(\frac{n\pi}{2}\right)\right\}_{n=0}^{\infty}, \ a_n = \left\{ \begin{array}{cc} 0 & n = 2k \text{ even,} \\ (-1)^k & n = 2k+1 \text{ odd.} \end{array} \right. .$$

- The sequence defined recursively by

  $$a_1 = \sqrt{2}, \ a_{n+1} = \sqrt{2 + a_n},$$

  converges to 2. The sequence is increasing by induction since $a_2 = \sqrt{2 + \sqrt{2}} \geq \sqrt{2} = a_1$ and

  $$a_{n+1} \geq a_n \Leftrightarrow \sqrt{2 + a_n} \geq \sqrt{2 + a_{n-1}} \Leftrightarrow a_n \geq a_{n-1}.$$

  The sequence is bounded $a_n < 2$ by induction since

  $$a_n < 2 \Leftrightarrow 2 + a_n < 4 \Leftrightarrow a_{n+1} = \sqrt{2 + a_n} < \sqrt{4} = 2.$$

  Hence the limit exists. If the limit is L, we have

  $$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + a_n} = \sqrt{2 + L}.$$

  Solving for L gives $L = 2$.

- For any real number t, the sequence $(1 + t/n)^n$ converges to the number e^t (see this week's quiz).

Determine whether or not the following limits exist and find the limit if it does exist.

1. $\dfrac{n!}{n^n}$

2. $n \sin(1/n)$

3. $\dfrac{n8^n}{3^{2n+1}}$

4. $\dfrac{\sin(n^2)}{\sqrt{n}}$

Write a formula for the sequences below:

1.

$$\frac{1}{2}, \frac{3}{5}, \frac{5}{8}, \frac{7}{11}, \frac{9}{14}, \ldots$$

2.

$$-\frac{1}{5}, \frac{1}{11}, -\frac{1}{29}, \frac{1}{83}, \ldots$$

3.

$$\frac{2}{1}, \frac{-8}{1 \cdot 2 \cdot 3}, \frac{32}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}, \frac{-128}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}, \ldots$$

1. Recall the sequence of Fibonacci numbers

$$F_1 = F_2 = 1, \; F_{n+1} = F_n + F_{n-1}, \; (1, 1, 2, 3, 5, 8, 11, \ldots).$$

Find $\lim_{n\to\infty} F_{n+1}/F_n$ (assuming it exists).

2. Considering the picture below, show that the sequence

$$a_N = \sum_{n=1}^{N-1} \frac{1}{n} - \int_1^N \frac{1}{x} dx$$

is increasing and bounded above by 1. The sequence therefore converges. (This number is known as Euler's constant γ.)