Ratio test

The ratio test compares a series to a geometric series. In a geometric series, the ratio of adjacent terms is constant

$$\frac{a_{n+1}}{a_n} = r$$

and the series converges if $|r| < 1$ (the terms of the series do not approach zero if $|r| \geq 1$). The ratio test is the following:

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ then

$$\sum_n a_n \begin{cases} \text{converges} & 0 \leq L < 1 \\ \text{diverges} & L > 1 \\ ??? & L = 1 \end{cases}.$$

For the ratio test to give convergence, the series must be converging as quickly as a convergent geometric series. If the ratio test gives divergence, then the terms of the series do not approach zero.

- Consider the series
$$\sum_{n=1}^{\infty} \frac{1}{n},\ \sum_{n=1}^{\infty} \frac{1}{n^2}.$$
The first diverges and the second converges but the ratio test says nothing about either:
$$\lim_{n\to\infty} \frac{1/(n+1)}{1/n} = 1 = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2}.$$

- The series
$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$
diverges since
$$\lim_{n\to\infty} \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e > 1.$$

- The series
$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$
converges since
$$\lim_{n\to\infty} \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim_{n\to\infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = 1/e < 1.$$

1. $\sum_{n=1}^{\infty}(-1)^n \frac{e^n}{n!}$

2. $\sum_{n=1}^{\infty} \frac{n^3 + 3n + 1}{3^n}$

3. $\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$

4. $\sum_{n=1}^{\infty} \frac{n^n}{e^n n!}$