MATH 2300-016 QUIZ 9 Name: ____________________________

For these problems, let $f(x) = (1+x)^{1/3}$

1. Find $f'(x)$, $f''(x)$, and $f'''(x)$.

   We have

$$
\begin{aligned}
f'(x) &= \left(\frac{1}{3}\right)(1+x)^{-2/3} \\
f''(x) &= \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)(1+x)^{-5/3} \\
f'''(x) &= \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)(1+x)^{-8/3}.
\end{aligned}
$$

2. What is the maximum M of $|f'''(x)|$ on the interval $[0,1]$?

   The function $|f'''(x)| = \frac{10}{27(1+x)^{7/3}}$ is decreasing on the interval $[0,1]$ hence attains its maximum at $x = 0$, $|f'''(0)| = \frac{10}{27} = M$.

3. What is $T_2(x)$, the second degree Taylor polynomial for f centered at $x = 0$?

   The second degree Taylor polynomial is

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 1 + \frac{x}{3} - \frac{x^2}{9}.$$

4. Use $T_2(x)$ to estimate $\sqrt[3]{2}$.

   With $x = 1$ we have $f(1) = \sqrt[3]{2}$ so we estimate using $T_2(1) = 1 + 1/3 - 1/9 = 11/9$.

5. The following theorem describes the difference between the nth Taylor polynomial of f and the function f itself.

Theorem. *Suppose f is $(n+1)$-times differentiable on an open interval containing a and x. Then there exists c between a and x with*

$$f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

where T_n is the nth Taylor polynomial of f centered at a.

In particular, taking absolute values and taking an upper bound

$$M \geq \max\{|f^{(n+1)}(t)| \ : \ t \text{ between } a \text{ and } x\},$$

we have the following inequality for the differnence between f and T_n:

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Bound the absolute value of the difference $f(1) - T_2(1) = \sqrt[3]{2} - T_2(1)$ using Taylor's inequality and the bound M on $|f'''(x)|$ you found above.

Taylor's inequality above (with $a = 0$, $x = 1$, $n = 2$, $f(x) = (1+x)^{1/3}$, and $M = 10/27$) states that

$$|\sqrt[3]{2} - 11/9| = |f(1) - T_2(1)| \leq \frac{M}{3!}|1-0|^3 = \frac{5}{81}.$$

Hence

$$\frac{94}{81} \leq \sqrt[3]{2} \leq \frac{104}{81}.$$