MATH 2300-016 QUIZ 6 Name: ____________________________

Collaborators (if any):

Due Monday, February 26th at the beginning of class. Submit your work on additional paper, treating this page as a cover sheet. You may use technology and work with with other students. If you work with others, please list their names above. SHOW YOUR WORK!

Determine whether the sequence converges or diverges. If it converges, find its limit.

1. $a_n = \dfrac{e^n + e^{-n}}{e^{2n} - 1}$

   Intuition: The denominator grows exponentially like $(e^2)^n$ while the numerator grows exponentially like e^n. Hence the quotient decays exponentially like $(1/e)^n$ and should go to zero as $n \to \infty$. To make this more rigorous, divide the numerator and denominator by e^{2n} to get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^{-n} + e^{-3n}}{1 - e^{-2n}} = \frac{0+0}{1+0} = 0.$$

2. $b_n = \ln(2n^2 + 1) - \ln(n^2 + 1)$

   Combing the logarithms and dividing the numerator and denominator of the argument by n^2 gives

$$b_n = \ln\left(\frac{2n^2+1}{n^2+1}\right) = \ln\left(\frac{2+1/n^2}{1+1/n^2}\right).$$

   Hence

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \ln\left(\frac{2+1/n^2}{1+1/n^2}\right) = \ln\left(\lim_{n\to\infty} \frac{2+1/n^2}{1+1/n^2}\right) = \ln\left(\frac{2+0}{1+0}\right) = \ln 2.$$

3. $c_n = \sqrt[n]{2^n + 3^n}$

   Intuition: 2^n is inconsequential when compared to 3^n, so the sequence behaves like $(3^n)^{1/n} = 3$. To make this rigorous, we factor out 3^n and take logarithms

$$\ln(c_n) = \frac{1}{n}\ln(3^n(1 + (2/3)^n)) = \ln 3 + \frac{1}{n}\ln(1 + (2/3)^n) \to \ln 3 + 0 = \ln 3.$$

   Hence $\lim_{n\to\infty} c_n = e^{\ln 3} = 3$.

4. $d_n = \dfrac{\sin(n)\ln n}{n}$

   Intuition: $\sin n$ is bounded by 1 and $\ln n$ grows much more slowly than n, so the sequence should converge to zero. To make this rigorous, we note that

$$|d_n| \le \frac{\ln n}{n}$$

and that

$$\lim_{x\to\infty}\frac{\ln x}{x} = \text{“}\frac{\infty}{\infty}\text{”} = \lim_{x\to\infty}\frac{1/x}{1} = 0$$

using L'Hôpital's rule. Hence

$$0 = -\lim_{n\to\infty}\frac{\ln n}{n} \le \lim_{n\to\infty} d_n \le \lim_{n\to\infty}\frac{\ln n}{n} = 0$$

and the limit is zero as expected.

5. $e_n = \left(1+\dfrac{t}{n}\right)^n$, where t is a constant.

   This sequence converges to e^t (and you should know this, perhaps taking it for a definition of e^t). Taking logarithms gives

   $$\lim_{n\to\infty}\ln e_n = \lim_{n\to\infty} n\ln(1+t/n) = \lim_{n\to\infty}\frac{\ln(1+t/n)}{1/n} = \text{“}\frac{0}{0}\text{”},$$

   an indeterminant form. We use L'Hôpital's rule

   $$\lim_{x\to\infty}\frac{\ln(1+t/x)}{1/x} = \lim_{x\to\infty}\frac{(-t/x^2)/(1+t/x)}{-1/x^2} = \lim_{x\to\infty}\frac{t}{1+t/x} = t,$$

   to conclude that

   $$\lim_{n\to\infty} e_n = e^t.$$

Determine whether the series is convergent or divergent. If it converges find the sum.

1. $\displaystyle\sum_{n=2}^{\infty}\frac{2}{n^2-1}$

   The series is telescoping. Partial fractions gives

   $$\begin{aligned}\sum_{n=2}^{\infty}\frac{2}{n^2-1} &= \sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n+1}\right) = \lim_{N\to\infty}\sum_{n=2}^{N}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)\\ &= \lim_{N\to\infty}(1-1/3)+(1/2-1/4)+(1/3-1/5)+\ldots+\left(\frac{1}{N-1}-\frac{1}{N+1}\right)\\ &= \lim_{N\to\infty} 1+1/2-\frac{1}{N}-\frac{1}{N+1} = 3/2.\end{aligned}$$

2. $\displaystyle\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k}\right)$

   The series is telescoping:

   $$\begin{aligned}\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k}\right) &= \sum_{k=1}^{\infty}\ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{\infty}(\ln(k+1)-\ln k)\\ &= \lim_{N\to\infty}(\ln 2-\ln 1)+(\ln 3-\ln 2)+\ldots+(\ln(N+1)-\ln N)\\ &= \lim_{N\to\infty}\ln(N+1) = \infty.\end{aligned}$$

   The limit of the partial sums does not exist, so the series does not converge.

3. $\displaystyle\sum_{m=1}^{\infty} \frac{m(m+2)}{(m+3)^2}$

   The terms of the series do not approach zero

   $$\lim_{m\to\infty} \frac{m(m+2)}{(m+3)^2} = 1,$$

   so the series cannot converge.

4. $\displaystyle\sum_{j=1}^{\infty} [(0.8)^{j-1} - (0.3)^j]$

   This is a sum of two convergent geometric series:

   $$\sum_{j=1}^{\infty} [(0.8)^{j-1} - (0.3)^j] = \sum_{j=1}^{\infty} (0.8)^{j-1} - \sum_{j=1}^{\infty} (0.3)^j = \frac{1}{1-0.8} - \frac{0.3}{1-0.3} = 38/7.$$

Show the following:

1. For any $\epsilon > 0$, $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x^\epsilon} = 0$. I.e., $\ln x$ grows more slowly than any power of x.

   The limit is indeterminant, ∞/∞. Applying L'Hôpital's rule gives

   $$\lim_{x\to\infty} \frac{\ln x}{x^\epsilon} = \lim_{x\to\infty} \frac{1/x}{\epsilon x^{\epsilon-1}} = \lim_{x\to\infty} \frac{1}{\epsilon x^\epsilon} = 0.$$

2. For any $p > 0$, $\displaystyle\lim_{x\to\infty} \frac{x^p}{e^x} = 0$. I.e., e^x (or a^x for any $a > 1$) grows more quickly than any power of x.

   Let n be the integer such that $n - 1 < p \le n$. Applying L'Hôpital's rule n times gives

   $$\lim_{x\to\infty} \frac{x^p}{e^x} = \lim_{x\to\infty} \frac{p(p-1)\cdot\ldots\cdot(p-n+1)x^{p-n}}{e^x} = \lim_{x\to\infty} \frac{p(p-1)\cdot\ldots\cdot(p-n+1)}{x^{n-p}e^x} = 0$$

   since $n - p \ge 0$.

Try these more challenging problems from the text.

1. [Section 8.2, problem 58]
   The lengths of the red line segments are

   $$b\sin\theta,\ b\sin^2\theta,\ \ldots,\ b\sin^n\theta,\ \ldots,$$

   the terms of a geometric series with first term$b\sin\theta$ and common ratio $\sin\theta$. Hence the total length is

   $$\frac{b\sin\theta}{1-\sin\theta}.$$

2. [Section 8.2, problem 68]
Answer: $11\pi/96$. Hint: If $(r_n)_{n=0}^{\infty}$ is the sequence of radii of the successively smaller circles, then the ratio r_{n+1}/r_n is constant.

The total area is

$$A = \pi r_0^2 + 3\sum_{n=1}^{\infty} \pi r_n^2,$$

adding the area of the middle circle and the areas of the three sequence of circles going towards the vertices. So we must determine the radii r_n.

Let $x = r_0$ be the length of the segment from the center of the triangle to one of the sides and perpendicular to that side, and let y be length of the segment from the center of the triangle to one of the vertices. Then

$$x + y = \sqrt{3}/2,\ x = y/2 \Rightarrow x = \frac{1}{2\sqrt{3}},\ y = 2x = \frac{1}{\sqrt{3}}.$$

Drawing a line parallel to the base of the triangle tangent to the top of the largest circle creates another equilateral triangle containing the rest of the circles going towards the top vertex. The smaller triangle is $1/3$ the size of the larger triangle since the height of the smaller triangle is x and the height of the large triangle is $3x$ (see figure). Hence

$$r_1 = r_0/3,\ r_2 = r_1/3,\ \ldots,\ r_{n+1} = r_n/3 \Rightarrow r_n = \frac{r_0}{3^n}.$$

Hence the total area is

$$\begin{aligned} A &= \pi r_0^2 + 3\sum_{n=1}^{\infty} \pi r_n^2 = \pi r_0^2 + 3\sum_{n=1}^{\infty} \pi (r_0/3^n)^2 = \pi r_0^2 + 3\pi r_0^2 \sum_{n=1}^{\infty} (1/9)^n \\ &= \pi r_0^2 \left(1 + 3\frac{1/9}{1-1/9}\right) = 11\pi/96. \end{aligned}$$