MATH 2300-016 QUIZ 2 Name: ____________________________

1. $\displaystyle\int \frac{x-1}{x^2-6x+9}dx$

The denominator factors as $x^2-6x+9=(x-3)^2$, so we want the partial fraction decomposition has the form

$$\frac{x-1}{x^2-6x+9}=\frac{A}{x-3}+\frac{B}{(x-3)^2}.$$

Clearing denominators gives

$$x-1=A(x-3)+B=Ax+(B-3A).$$

Equating the coefficients of powers of x give a system of linear equations

$$\begin{aligned}1&=A\\ -1&=B-3A,\end{aligned}$$

which has solution $A=1$, $B=2$. Hence

$$\int \frac{x-1}{x^2-6x+9}dx=\int\frac{1}{x-3}dx+\int\frac{2}{(x-3)^2}dx=\ln|x-3|-\frac{2}{x-3}+C.$$

2. $\displaystyle\int \frac{3x+1}{x^2+4}dx$

We have

$$\int\frac{3x+1}{x^2+4}dx=\int\frac{3x}{x^2+4}+\int\frac{1}{x^2+4}dx.$$

Making the substitution $u=x^2+4$, $du=2xdx$ in the first integral and recalling $\int\frac{dx}{x^2+a^2}=\frac{1}{a}\arctan(x/a)$ for the second gives

$$\int\frac{3x}{x^2+4}+\int\frac{1}{x^2+4}dx=\frac{3}{2}\int\frac{du}{u}+\frac{1}{2}\arctan(x/2)=\frac{3}{2}\ln|x^2+4|+\frac{1}{2}\arctan(x/2)+C.$$

3. $\displaystyle\int \frac{x^2}{(1-x^2)^{3/2}}dx$

With the substitution $x=\sin\theta$, $dx=\cos\theta\ d\theta$, we get

$$\begin{aligned}\int\frac{x^2}{(1-x^2)^{3/2}}dx&=\int\frac{\sin^2\theta\cos\theta\ d\theta}{(1-\sin^2\theta)^{3/2}}=\int\frac{\sin^2\theta\cos\theta\ d\theta}{(\cos^2\theta)^{3/2}}=\int\frac{\sin^2\theta}{\cos^2\theta}d\theta\\ &=\int\tan^2\theta\ d\theta=\int(\sec^2\theta-1)d\theta=\tan\theta-\theta+C.\end{aligned}$$

Our substituion is $\theta=\arcsin x$ so that $\tan\theta=\tan(\arcsin x)=\frac{x}{\sqrt{1-x^2}}$ (draw a triangle). Hence

$$\int\frac{x^2}{(1-x^2)^{3/2}}dx=\frac{x}{\sqrt{1-x^2}}-\arcsin x+C.$$

4. $\int \ln(\sin x) \sin x \cos x \; dx$ (Make a substitution then integrate by parts.)
With $y = \sin x$, $dy = \cos x \; dx$ we have

$$\int \ln(\sin x) \sin x \cos x \; dx = \int y \ln y \; dy.$$

Now integrate by parts with

$$\begin{aligned} u &= \ln y, \quad & du &= dy/y, \\ dv &= y, \quad & v &= y^2/2, \end{aligned}$$

to get

$$\int y \ln y \; dy = \frac{y^2}{2} \ln y - \frac{1}{2} \int y \; dy = \frac{y^2}{2} \ln y - \frac{y^2}{4}.$$

Hence

$$\int \ln(\sin x) \sin x \cos x \; dx = \frac{\sin^2 x}{2} \ln(\sin x) - \frac{\sin^2 x}{4} + C = \frac{\sin^2 x}{2}(\ln(\sin x) - 1/2) + C.$$

5. (a) Divide: $\dfrac{3x^2 + 2x + 1}{x + 2} = Ax + B + \dfrac{C}{x + 2}$.

$$\frac{3x^2 + 2x + 1}{x + 2} = 3x - 4 + \frac{9}{x + 2}$$

(b) Complete the square: $x^2 + x + 1 = (x + A)^2 + B$.

$$x^2 + x + 1 = (x + 1/2)^2 + 3/4.$$

(c) Find the exact value of $\sin(\pi/8)$ using the half-angle formula.
We have

$$\sin^2(\pi/8) = \frac{1 - \cos(\pi/4)}{2} = \frac{1 - 1/\sqrt{2}}{2} = \frac{2 - \sqrt{2}}{4}.$$

Hence

$$\sin(\pi/8) = \frac{\sqrt{2 - \sqrt{2}}}{2}.$$