MATH 2300-016 QUIZ 14/15 Name: ______________________

Collaborators (if any):

Due Wednesday, May 2nd at the beginning of class. Submit your work on additional paper, treating this page as a cover sheet. You may use technology and work with with other students. If you work with others, please list their names above. The first two problems are essential, while the last two are extra-cirricular.

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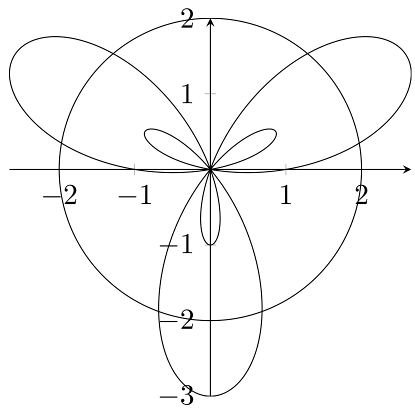
1. The polar curves

$$r(\theta) = 1 + 2\sin(3\theta),\ r = 2,$$

are graphed below.



(a) Find the area inside the larger loops and outside the smaller loops of the graph of $r = 1 + 2\sin(3\theta)$. [Hint: Use symmetry, the answer is $\pi + 3\sqrt{3}$.]

Solution. Solving $r = 0$ gives

$$r = 1{+}2\sin(3\theta) = 0,\ \sin(3\theta) = -1/2,\ 3\theta = -\pi/6, 7\pi/6{+}2\pi k,\ \theta = -\pi/18, 7\pi/18{+}2\pi k/3.$$

One of the "big" loops is traced out by $-\pi/18 \le \theta \le 7\pi/18$, and one of the small loops is traced out by $7\pi/18 \le \theta \le 11\pi/18$. The corresponding areas are

$$\begin{aligned}
A_{big} &= \frac{1}{2}\int_{-\pi/18}^{7\pi/18} (1 + 2\sin(3\theta))^2 d\theta = \frac{1}{2}\int_{-\pi/18}^{7\pi/18} (1 + 4\sin(3\theta) + 4\sin^2(3\theta))d\theta \\
&= \frac{1}{2}\int_{-\pi/18}^{7\pi/18} (1 + 4\sin(3\theta) + 2(1 - \cos(6\theta))d\theta \\
&= \frac{1}{2}\left(3\theta - \frac{4}{3}\cos(3\theta) - \frac{1}{3}\sin(6\theta)\right)\Big|_{-\pi/18}^{7\pi/18} \\
&= \frac{\sqrt{3}}{2} + \frac{2\pi}{3} = 2.96042\ldots,
\end{aligned}$$

$$
\begin{aligned}
A_{small} &= \frac{1}{2}\int_{7\pi/18}^{11\pi/18}(1+2\sin(3\theta))^2 d\theta = \frac{1}{2}\int_{7\pi/18}^{11\pi/18}(1+4\sin(3\theta)+4\sin^2(3\theta))d\theta \\
&= \frac{1}{2}\int_{7\pi/18}^{11\pi/18}(1+4\sin(3\theta)+2(1-\cos(6\theta))d\theta \\
&= \frac{1}{2}\left(3\theta - \frac{4}{3}\cos(3\theta) - \frac{1}{3}\sin(6\theta)\right)\Big|_{7\pi/18}^{11\pi/18} \\
&= \frac{\pi}{3} - \frac{\sqrt{3}}{2} = 0.18117\ldots.
\end{aligned}
$$

Hence the total area inside the big loops and outside the small loops is

$$3(A_{big} - A_{small}) = \pi + 3\sqrt{3} = 8.3377\ldots.$$

(b) Find the area outside the circle $r = 2$ but inside the curve $r = 1 + 2\sin(3\theta)$.
[Answer: $\frac{5\sqrt{3}}{2} - \frac{\pi}{3}$.]
Solution. We have

$r = 2 = 1+2\sin(3\theta),\ \sin(3\theta) = 1/2,\ 3\theta = \pi/6, 5\pi/6+2\pi k,\ \theta = \pi/18, 5\pi/18+2\pi k/3.$

One third of the area outside the circle and inside the other curve is given by

$$
\begin{aligned}
\frac{A}{3} &= \frac{1}{2}\int_{\pi/18}^{5\pi/18}[(1+2\sin(3\theta))^2 - 2^2]d\theta = \frac{1}{2}\int_{\pi/18}^{5\pi/18}(-3+4\sin(3\theta)+4\sin^2(3\theta))d\theta \\
&= \int_{\pi/18}^{5\pi/18}(-3/2+2\sin(3\theta)+1-\cos(6\theta))d\theta \\
&= \left(-\frac{\theta}{2} - \frac{2}{3}\cos(3\theta) - \frac{1}{6}\sin(6\theta)\right)\Big|_{\pi/18}^{5\pi/18} \\
&= \frac{5}{2\sqrt{3}} - \frac{\pi}{9} = 1.09430\ldots.
\end{aligned}
$$

Hence the total area outside the circle but inside the other curve is

$$A = \frac{5\sqrt{3}}{2} - \frac{\pi}{3} = 3.28292\ldots.$$

(c) What is the tangent line to the curve $r = 1 + 2\sin(3\theta)$ at the point in the first quadrant where r is maximum?
Solution. The maximal value of $r = 1 + 2\sin(3\theta)$ is $r(\pi/6) = 3$, which happens at $(x, y) = (3\cos(\pi/6), 3\sin(\pi/6)) = (3\sqrt{3}/2, 3/2)$. Generally, we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{r\cos\theta + \sin\theta\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\frac{dr}{d\theta}}$$

which with $r = 1 + 2\sin(3\theta)$ gives

$$\frac{dy}{dx} = \frac{(1+2\sin(3\theta))\cos\theta + \sin\theta(6\cos(3\theta))}{-(1+2\sin(3\theta))\sin\theta + \cos\theta(6\cos(3\theta))}.$$

At $\theta = \pi/6$ we have

$$\frac{dy}{dx}\Big|_{\theta=\pi/6} = -\sqrt{3}.$$

Hence the tangent line is

$$y - 3/2 = -\sqrt{3}(x - 3\sqrt{3}/2).$$

(d) Write down a definite integral for the arclength of the curve $r(\theta) = 1 + 2\sin(3\theta)$ and use a computer to evaluate. [Answer: $27.2667\ldots$.]

Solution. The arclength of a parametric curve $(x(t), y(t))$ for $a \leq t \leq b$ is given by

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt.$$

Taking $r = r(\theta)$, $(x(\theta), y(\theta)) = (r\cos\theta, r\sin\theta)$ (the curve is parameterized by θ), we get

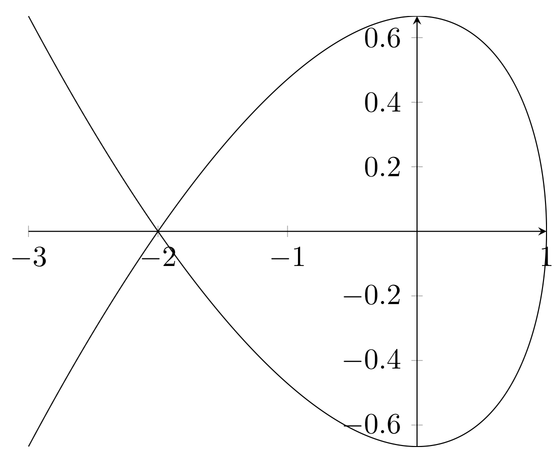
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

In the case at hand, we have

$$L = \int_0^{2\pi} \sqrt{(1 + 2\sin(3\theta))^2 + (6\cos(3\theta))^2}d\theta = 27.2667\ldots.$$

2. Consider the parametric curve defined by

$$x(t) = 1 - t^2,\ y(t) = t - t^3/3.$$



(a) Find the equations of the tangent lines to the curve at the point $(-2, 0)$.

Solution. The slope of the tangent line at the point $(x(t), y(t))$ is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - t^2}{-2t}.$$

The t-values corresponding the the point $(-2, 0)$ are $t = \pm\sqrt{3}$ and the corresponding slopes are

$$\left.\frac{dy}{dx}\right|_{t=\pm\sqrt{3}} = \frac{-2}{-2(\pm\sqrt{3})} = \pm\frac{1}{\sqrt{3}}.$$

Hence the tangent lines are

$$y = \frac{x+2}{\sqrt{3}}\ (t = \sqrt{3}),\ y = \frac{-(x+2)}{\sqrt{3}}\ (t = -\sqrt{3}).$$

(b) When/where does the curve have horizontal tangents?

Solution. The slope dy/dx is zero when $dy/dt = 1 - t^2 = 0$, i.e. when $t = \pm 1$. This corresponds to $(x, y) = (0, \pm 2/3)$.

(c) What is the length of the part of the curve forming the "loop"? [Answer: $4\sqrt{3}$.]

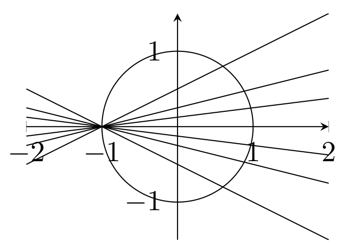
Solution. The t-values of the point of self-intersection $(-2, 0)$ were found to be $t = \pm\sqrt{3}$ above. The arclength is

$$\begin{aligned} L &= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(-2t)^2 + (1-t^2)^2}dt = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4t^2 + 1 - 2t^2 + t^4}dt \\ &= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(1+t^2)^2}dt = \int_{-\sqrt{3}}^{\sqrt{3}} (1+t^2)dt = t + t^3/3\Big|_{-\sqrt{3}}^{\sqrt{3}} \\ &= 4\sqrt{3} = 6.9282\ldots. \end{aligned}$$

3. Find the other point of intersection of the line with slope t going through the point $(-1, 0)$ and the circle of radius one,

$$y = t(x+1),\ x^2 + y^2 = 1,$$

as a function of t. [This gives a rational parameterization of the unit circle, $(x(t), y(t)) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$.] Writing $t = m/n$, clear denominators in $x(t)^2 + y(t)^2 = 1$ to give a parameterization of Pythagorean triples, integers (a, b, c) with $a^2 + b^2 = c^2$.



Solution. Substituting the equation of the line into the equation for the circle, we have

$$1 = x^2 + y^2 = x^2 + (t(x+1))^2,\ (1+t^2)x^2 + (2t^2)x + t^2 - 1 = 0,$$

$$x = \frac{-2t^2 \pm \sqrt{(2t^2)^2 - 4(1+t^2)(t^2-1)}}{2(1+t^2)} = -1,\ \frac{1-t^2}{1+t^2}.$$

Putting these values of x into the equation for the line gives

$$y = 0,\ \frac{2t}{1+t^2}.$$

Hence the other point of intersection, which depends on the slope t, is

$$(x(t), y(t)) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right).$$

If $t = m/n$ is a rational number, we get

$$x^2 + y^2 = 1 \Longrightarrow \left(\frac{1-(m/n)^2}{1+(m/n)^2}\right)^2 + \left(\frac{2(m/n)}{1+(m/n)^2}\right)^2 = 1,$$

and clearing denominators gives

$$(n^2+m^2)^2 = (n^2-m^2)^2 + (2mn)^2.$$

E.g. $m=1$, $n=2$ gives

$$5^2 = 3^2 + 4^2,$$

or $m=2$, $n=3$ gives

$$13^2 = 5^2 + 12^2.$$

In fact, every integer solution to $a^2+b^2=c^2$ comes from the above for some choice of m and n.

4. This problem explores the relationship between polar coordinates and complex numbers.

   - For $\theta \in \mathbb{R}$, use the power series for e^z to show that

     $$e^{i\theta} = \cos\theta + i\sin\theta$$

     where $i^2=-1$ (at least formally, since we won't discuss convergence of series of complex numbers). [Hence polar coordinates $(x,y)=(r\cos\theta, r\sin\theta)$ can be written in terms of complex numbers as

     $$x+iy = r\cos\theta + ir\sin\theta = re^{i\theta},$$

     with x and y are the real and imaginary part of $re^{i\theta}$.]
     Solution. We have

     $$e^{i\theta} = \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!} = \sum_{n=2k \text{ even}}\frac{(-1)^k\theta^{2k}}{(2k)!} + i\sum_{n=2k+1 \text{ odd}}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!} = \cos\theta + i\sin\theta.$$

   - Show that the exponential identity

     $$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$$

     is equivalent to the trigonometric identities

     $$\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi),\ \sin(\theta+\phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi).$$

     [In particular, to multiply two complex numbers

     $$z = x+iy = re^{i\theta},\ w = u+iv = se^{i\phi},$$

     we multiply their lengths r, s, and add their angles θ, ϕ:

     $$zw = (xu-yv) + i(xv+yu) = rse^{i(\theta+\phi)}.$$

     Solution. We have

     $$\begin{aligned}\cos(\theta+\phi) + i\sin(\theta+\phi) &= e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\\ &= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi),\end{aligned}$$

     use multiplication of complex numbers and properties of the exponential function.