MATH 2300-016 QUIZ 13 Due Tuesday, April 24th Name: ______________________

1. Recall Newton's law of cooling: the rate of change in temperature of an object is proportional to the difference in temperature between the object and its surroundings,

$$\frac{dT}{dt} = k(T - T_s),$$

where $T(t)$ is temperature as a function of time, k is the proportionality constant, and T_s is the constant surrounding temperature.

Suppose a cup of coffee is 200°F when it is poured and has cooled to 190°F after one minute in a room at 70°F. When will the coffee reach 150°F? What will the temperature of the coffee be after it sits for 30 minutes?

Solution. First of all, the general solution to the differential equation above is

$$T(t) = T_s + Ce^{-kt},\ C = T(0) - T_s.$$

The information $T(1) = 190$ (time in minutes) gives us the value of k:

$$190 = T(1) = 70 + (200 - 70)e^{k\cdot 1},\ k = -\ln(12/13) \approx 0.08.$$

We now know everything about T, for instance

$$T(t) = 150 = 70 + 130e^{-kt},\ t = -\ln(8/13)/k \approx 6.065 \text{ minutes},$$

and

$$T(30) = 70 + 130e^{-k\cdot 30} \approx 81.7°\text{F}.$$

2. The following variation on the logistic equation models logistic growth with constant harvesting:

$$\frac{dP}{dt} = kP(1 - P/M) - c.$$

For this problem consider the specific instance

$$\frac{dP}{dt} = 0.08P(1 - P/1000) - 15,$$

modeling fish population in a pond where 15 fish per week are caught (time t in weeks).

(a) What are the equilibrium solutions to the differential equation in part (i.e. what are the constant solutions)?

Solution. The equilibrium solutions are the zeros of the right-hand side of the differential equation, $P = 750,\ 250$.

(b) Find the general solution of the differential equation. [Integrate using partial fractions. You should get something equivalent to $P(t) = \dfrac{750 - 250Ce^{-t/25}}{1 - Ce^{-t/25}}$ where C is an arbitrary constant.]

Solution. Separating variables, multiplying by a constant, and integrating gives

$$\int \frac{dP}{P^2 - 1000P + 187500} = -\int \frac{dt}{12500}.$$

Partial fractions on the left-hand side gives

$$\frac{1}{P^2 - 1000P + 187500} = \frac{1}{(P-750)(P-250)} = \frac{1/500}{P-750} + \frac{-1/500}{P-250}.$$

Integrating, we obtain

$$\ln\left|\frac{P-750}{P-250}\right| = -\frac{t}{25} + C.$$

Exponentiating gives (different C)

$$\frac{P-750}{P-250} = Ce^{-t/25}$$

Finally, solving for P gives the general solution

$$P(t) = \frac{750 - 250Ce^{-t/25}}{1 - Ce^{-t/25}}$$

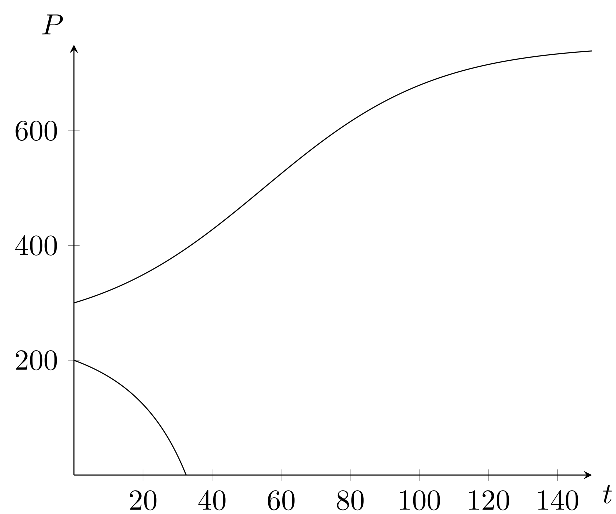
(c) Find and interpret the solutions with initial conditions $P(0) = 200, 300$.
Solution. We need to solve for C in the following

$$200 = \frac{750 - 250C}{1 - C}, \; 300 = \frac{750 - 250C}{1 - C}$$

obtaining $C = 11$ and $C = -9$ respectively. So the two particular solutions are

$$P(t) = \frac{750 - 2750e^{-t/25}}{1 - 11e^{-t/25}}, \; P(t) = \frac{750 + 2250e^{-t/25}}{1 + 9e^{-t/25}}.$$

In the first solution ($P(0) = 200$), the population reaches zero at $t = 25\ln(11/3) \approx 32.48$ weeks, i.e. fishing at a rate of 15 fish/week is unsustainable, while in the second solution ($P(0) = 300$), as $t \to \infty$ the population approaches 750 and the population can sustain this level of fishing.



Here's a "heads-up" concerning material for the rest of the semester. You will definitely be responsible for the material in sections 1.7, 3.4, and 6.4 concerning parametric curves, slopes/tangents to parametric curves, and arclength of parametric curves. [You may also want to look at sections 9.1. 9.2, and 9.5 concerning 3D coordinates and vectors, and 10.1, 10.2, 10.3 concerning some simple vector calculus - this is mostly optional.] We will also be covering the material in appendices H.1 and H.2, polar coordinates and arclength/area in polar coordinates.