MATH 2300-016 QUIZ 11 (take home, due Monday 4/9) Name: ____________________

1. Express
$$\int_0^1 \sin(x^2)dx$$
as an infinite series. Use the first two terms of the series to approximate the definite integral and bound the error using the alternating series remainder estimate.

Since $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x, we have

$$\int_0^1 \sin(x^2)dx = \int_0^1 \sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!}dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}\Big|_0^1$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!} = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \cdots.$$

This is a convergent alternating series. Using the first two terms to estimate the definite integral (i.e. the $n = 0$ and 1 terms), we get

$$\left|\int_0^1 \sin(x^2)dx - \left(\frac{1}{3} - \frac{1}{42}\right)\right| \le \frac{1}{(4(2)+3)(2(2)+1)!} = \frac{1}{1320} = 0.00\overline{75},$$

or

$$0.30876\ldots = \frac{951}{3080} = \frac{13}{42} - \frac{1}{1320} \le \int_0^1 \sin(x^2) \le \frac{13}{42} + \frac{1}{1320} = \frac{2867}{9240} = 0.31028\ldots.$$

2. Starting with the geometric series, $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ for $|x| < 1$, show that
$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$
[Hint: Evaluate $\arctan(x)$ at $1/\sqrt{3}$.]

For $|x| < 1$, we have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n},$$
$$\arctan x = \int_0^t \frac{dt}{1+t^2} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}.$$

Evaluating both sides at $x = 1/\sqrt{3}$, we get

$$\frac{\pi}{6} = \arctan(1/\sqrt{3}) = \sum_{n=0}^{\infty}(-1)^n \frac{(1/\sqrt{3})^{2n+1}}{2n+1} = \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)},$$

which gives the result after multiplying both sides by 6.

3. Find the Taylor series for $\cos x$ centered at $x = \pi/3$ in two ways: from scratch and using the trigonometric identity

$$\cos(x) = \cos(x - \pi/3 + \pi/3) = \cos(x - \pi/3)\cos(\pi/3) - \sin(x - \pi/3)\sin(\pi/3)$$

along with knowledge of the Taylor series for $\sin x$, $\cos x$ centered at $x = 0$.

Let $f(x) = \cos x$. The derivatives of f are periodic, with values at $a = \pi/3$ as follows (here $n = 4k + l$):

l	$f^{(n)}(x)$	$f^{(n)}(\pi/3)$
0	$\cos x$	$1/2$
1	$-\sin x$	$-\sqrt{3}/2$
2	$-\cos x$	$-1/2$
3	$\sin x$	$\sqrt{3}/2$

So the Taylor series for f centered at $a = \pi/3$ is (separating into even and odd n)

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(x-\pi/3)^{2n}}{(2n)!} - \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(x-\pi/3)^{2n+1}}{(2n+1)!}.$$

Or we can use the trigonometric identity above and evaluate the Taylor series for sine and cosine at $x - \pi/3$:

$$\begin{aligned}
\cos(x) &= \cos(x - \pi/3 + \pi/3) = \cos(x - \pi/3)\cos(\pi/3) - \sin(x - \pi/3)\sin(\pi/3) \\
&= \cos(\pi/3)\sum_{n=0}^{\infty}\frac{(-1)^n(x-\pi/3)^{2n}}{(2n)!} - \sin(\pi/3)\sum_{n=0}^{\infty}\frac{(-1)^n(x-\pi/3)^{2n+1}}{(2n+1)!} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(x-\pi/3)^{2n}}{(2n)!} - \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(x-\pi/3)^{2n+1}}{(2n+1)!}.
\end{aligned}$$

4. Show that

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

for all x, i.e. use Taylor's inequality to show that the difference between $\cos x$ and its ($2N$th degree) Taylor polynomial

$$\cos x - \sum_{n=0}^{N} (-1)^n \frac{x^{2n}}{(2n)!}$$

goes to zero as $N \to \infty$.

Fix a value of x. For any n the abolute value of the $(n+1)$st derivative of $\cos t$ is bounded above by $M = 1$ on the interval between zero and x (because the $(n+1)$st derivative is one of $\cos t$, $-\sin t$, $-\cos t$, or $\sin t$, all of which are bounded between ± 1). Taylor's inequality shows that

$$|\cos x - T_n(x)| \leq \frac{M}{(n+1)!}|x-0|^{n+1} = \frac{x^{n+1}}{(n+1)!}.$$

Taking limits as $n \to \infty$, we get

$$\left| \cos x - \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \right| = \lim_{n \to \infty} |\cos x - T_n(x)| \leq \lim_{n \to \infty} \frac{x^{n+1}}{(n+1)!} = 0,$$

so that $\cos x$ is equal to its Taylor series centered at zero.

5. Find $T_4(x)$, the fourth degree Taylor polynomial for $f(x) = \sqrt{x}$ centered at $x = 9$, and use $T_4(x)$ to estimate $\sqrt{10}$. Use Taylor's inequality to bound the error in the approximation.

We have

$$f'(x) = \frac{1}{2}x^{-1/2},\ f''(x) = \frac{-1}{4}x^{-3/2},\ f'''(x) = \frac{3}{8}x^{-5/2},\ f^{(4)}(x) = \frac{-15}{16}x^{-7/2},\ f^{(5)}(x) = \frac{105}{32}x^{-9/2}$$

and

$$f(9) = 3,\ f'(9) = \frac{1}{6},\ f''(9) = \frac{-1}{108},\ f'''(9) = \frac{1}{648},\ f^{(4)}(9) = \frac{-5}{11664}.$$

Hence the fourth degree Taylor polynomial centered at $a = 9$ is

$$T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}(9)}{n!}(x-9)^n = 3 + \frac{(x-9)}{6} - \frac{(x-9)^2}{216} + \frac{(x-9)^3}{3888} - \frac{5(x-9)^4}{279936}.$$

This give an approximation of

$$\sqrt{10} = f(10) \approx T_4(10) = \frac{885235}{279936} = 3.162276\ldots$$

The maximum of $|f^{(5)}(x)|$ on the interval $[9, 10]$ is at $x = 9$

$$M = |f^{(5)}(9)| = \frac{35}{209952}.$$

Hence

$$|f(10) - T_4(10)| \le \frac{M}{(4+1)!}|10-9|^5 = \frac{7}{5038848} \approx 1.3892 \times 10^{-6}$$

or

$$3.1622749\ldots = \frac{15934223}{5030848} \le \sqrt{10} \le \frac{15934237}{5038848} = 3.1622777\ldots,$$

good to six digits.