MATH 2300-016 QUIZ 10 (in class/take home average) Name: ____________________

1. [Memorization] What are the Taylor series for the following functions (centered at zero)?

   (a) $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

   (b) $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

   (c) $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

   (d) $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

   (e) $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$

2. Consider the power series

$$\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}(x-1)^n$$

(a) Where is the power series centered?

Centered at $a = 1$

(b) What is the radius of convergence of the power series?

Using the ratio test, we have

$$\lim_{n\to\infty} \frac{|x-1|^{n+1} 2^{n+1}/\sqrt{n+1}}{|x-1|^n 2^n/\sqrt{n}} = 2|x-1| \lim_{n\to\infty} \sqrt{1+1/n} = 2|x-1|.$$

So the series converges for $|x-1| < 1/2$ and diverges for $|x-1| > 1/2$ and the radius of convergence is $1/2$.

(c) What is the interval of convergence of the power series?

We need to check the endpoints $x = 1/2,\ 3/2$. We have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}, \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

at $x = 1/2,\ 3/2$ respectively. The first series converges by the alternating series test and the second is a divergent p-series, $p = 1/2 < 1$. Hence the interval of convergence is $[1/2, 3/2)$.

3. Evaluate the limit

$$\lim_{x\to 0} \frac{1 + x^2 + x^4/2 - e^{x^2}}{x^6}.$$

We have

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = 1 = x^2 + x^4/2 + x^6/6 + x^8/24 + \ldots,$$

so that

$$\lim_{x\to 0} \frac{1 + x^2 + x^4/2 - e^{x^2}}{x^6} = \lim_{x\to 0} \frac{1 + x^2 + x^4/2 - \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}}{x^6}$$
$$= \lim_{x\to 0} \frac{1 + x^2 + x^4/2 - (1 + x^2 + x^4/2 + x^6/6 + \ldots)}{x^6} = \lim_{x\to 0} \frac{-\sum_{n=3}^{\infty} x^{2n}/n!}{x^6}$$
$$= -\lim_{x\to 0} \sum_{n=3}^{\infty} \frac{x^{2n-6}}{n!} = \lim_{x\to 0} -1/6 - x^2/4! - x^4/5! - \ldots = -1/6.$$

You could also use L'Hôpital's rule six times to get the same answer.

4. Suppose the power series $\sum_{n=0}^{\infty} c_n(x-1)^n$ converges at $x = 3$ and diverges at $x = -3$.

What can you say about the following series?

The above information tells us that $2 \leq R \leq 4$.

(a) $\sum_{n=0}^{\infty} \frac{c_n}{2^n}(-1)^n$ Converges / Diverges / Not enough information

Converges since $x = 1/2$ is inside the radius of convergence, $|1/2 - 1| < 2$.

(b) $\sum_{n=0}^{\infty} c_n 4^n$ Converges / Diverges / Not enough information

Not enough information since $x = 5$ could be outside the radius of convergence (at worst) or possibly an endpoint of the interval of convergence (at best),
$4 \geq |5 - 1| > 2$.

(c) $\sum_{n=0}^{\infty} c_n(-1)^n$ Converges / Diverges / Not enough information

Converges since $x = 0$ is inside the radius of convergence, $|0 - 1| < 2$

(d) $\sum_{n=0}^{\infty} c_n 6^n$ Converges / Diverges / Not enough information

Diverges since $x = 7$ is outside the radius of convergence, $|7 - 1| > 4$.

5. For this problem let $f(x) = 1 + xe^x$.

(a) What is the third degree Taylor polynomial centered at zero for $f(x)$?
Using the Taylor series for e^x, we have

$$1 + xe^x = 1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = 1 + x + x^2 + x^3/2 + x^4/3! + x^5/4! + \ldots.$$

Truncating this at degree three gives

$$T_3(x) = 1 + x + x^2 + x^3/2.$$

Or we can differentiate and plug in zero to get:

$$f(x) = 1 + xe^x,\ f(0) = 1,\ f'(x) = (x+1)e^x,\ f'(0) = 1,$$

$$f''(x) = (x+2)e^x,\ f''(0) = 2,\ f'''(x) = (x+3)e^x,\ f'''(0) = 3,$$

so that the third degree Taylor polynomial is

$$T_3(x) = \frac{f(0)}{0!}(x-0)^0 + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 = 1+x+x^2+x^3/3.$$

(b) What is the Taylor series for $f(x)$ centered at zero?
Using the Taylor series for e^x, we have

$$1 + xe^x = 1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = 1 + x + x^2 + x^3/2 + x^4/3! + x^5/4! + \ldots.$$

Or, from the definition of the coefficients for the Taylor series

$$c_n = \frac{f^{(n)}(a)}{n!}$$

and the easily established pattern

$$f^{(n)}(x) = (x+n)e^x,\ n \geq 1,$$

we get the same answer.