Power series

A power series (centered at a) is a function of the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \ c_n \in \mathbb{R},$$

a "polynomial of infinite degree." They behave somewhat like geometric series in that there is some $0 \leq R \leq \infty$, the **radius of convergence**, such that the series $f(x)$ converges for $|x-a| < R$ and diverges for $|x-a| > R$. Convergence at the end points of the the interval $[a-R, a+R]$ must be checked separately. The **interval of convergence** is (the largest interval) where the power series converges, one of

$$\begin{aligned} R = 0 &: \{a\}, \\ R = \infty &: (-\infty, \infty), \\ 0 < R < \infty &: (a-R, a+R),\ (a-R, a+R],\ [a-R, a+R),\ [a-R, a+R]. \end{aligned}$$

The radius of convergence is determined by

$$\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{1/n}$$

although we won't discuss this concept ($\limsup_{n\to\infty} a_n$ is the largest limit of any subsequence of a_n), but if the limits

$$\lim_{n\to\infty} |c_n|^{1/n} \text{ or } \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$$

exist, then they are equal to each other and to $1/R$. In other words, we can often use the ratio test or root test to find the radius of convergence. The convergence of the series inside the radius of convergence (i.e. on the interval $(a-R, a+R)$) is "uniform" (a techinical concept that says that the rate of convergence of the partial sums to $f(x)$ doesn't depend on x in any closed subinterval inside the radius of convergence), but it implies the following:

- [term-by-term differentiation] The function $f(x)$ is differentiable on $(a-R, a+R)$ and its derivative is given by

  $$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1},$$

  i.e. we can interchange Σ and $\frac{d}{dx}$. The radius of convergence of the resulting power series $f'(x)$ is also R.

- [term-by-term integration] The function $f(x)$ is integrable on $(a-R, a+R)$ and an antiderivative is given by

  $$\int f(x)dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1},$$

  i.e. we can interchange Σ and $\int$. The radius of convergence of the resulting power series $\int f(x)\ dx$ is also R.

Find the interval of convergence of the following power series:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n}(x-1)^n$

   The ratio test gives convergence

   $$\lim_{n\to\infty} \frac{3^{n+1}|x-1|^{n+1}/(n+1)}{3^n|x-1|^n/n} = 3|x-1| \lim_{n\to\infty} \frac{n}{n+1} = 3|x-1| < 1$$

   so the radius of convergence is $R = 1/3$. At the endpoints $x = 2/3, 4/3$ we have the series

   $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \ \sum_{n=1}^{\infty} \frac{1}{n}$$

   respectively (the first converges by the AST and the second is the divergent harmonic series). Hence the interval of convergence is $[2/3, 4/3)$.

2. $\displaystyle\sum_{n=1}^{\infty} n^n (x-2)^n$

   The ratio test gives

   $$\lim_{n\to\infty} \frac{(n+1)^{n+1}|x-2|^{n+1}}{n^n|x-2|^n} = |x-2| \lim_{n\to\infty} (1+1/n)^n(n+1) = \infty$$

   unless $x = 2$. Hence $R = 0$ and the interval of convergence is $\{2\}$ (the series only converges at $x = 2$).

3. $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{n!}(x-3)^n$

   The ratio test gives

   $$\lim_{n\to\infty} \frac{e^{n+1}|x-3|^{n+1}/(n+1)!}{e^n|x-3|^n/n!} = e|x-1| \lim_{n\to\infty} \frac{1}{n+1} = 0$$

   so the radius of convergence is $R = \infty$. Hence the interval of convergence is $(-\infty, \infty)$ (i.e. the series converges everywhere).

4. $\displaystyle\sum_{n=2}^{\infty} \frac{(x-4)^n}{n(\ln n)^{3/2}}$

The ratio test gives convergence when

$$\lim_{n\to\infty} \frac{|x-4|^{n+1}/(n+1)\ln(n+1)^{3/2}}{|x-4|^n/n(\ln n)^{3/2}} = |x-4| \lim_{n\to\infty} \frac{n}{n+1}\left(\frac{\ln n}{\ln(n+1)}\right)^{3/2} = |x-4| < 1$$

so the radius of convergence is $R = 1$. At the endpoints $x = 3, 5$ we have the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(\ln n)^{3/2}}, \quad \sum_{n=1}^{\infty} \frac{1}{n(\ln n)^{3/2}}$$

respectively (the first converges by the AST and the second is the convergent by the integral test). Hence the interval of convergence is $[3, 5]$.

An example we've seen before is the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad -1 < x < 1.$$

This power series is centered at $a = 0$ and has radius of convergence $R = 1$. Let's manipulate this series using the properties above to find power series representations for some other functions.

1. Integrate $\displaystyle\sum_{n=0}^{\infty} x^n$ to find a power series representation for the function $\ln(1-x)$ on the interval $(-1, 1)$. What is the interval of convergence for the resulting power series?

We have

$$-\ln(1-x) + C = \int \frac{dx}{1-x} = \int \left(\sum_{n=0}^{\infty} x^n\right) dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Evaluating both sides at $x = 0$ shows that $C = 0$. Hence

$$\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for} \ -1 < x < 1.$$

Note that the radius of convergence for the new series is 1, the same as the geometric series. However, the new series also converges at $x = -1$.

2. Differentiate $\sum_{n=0}^{\infty} x^n$ term-by-term to find a power series representation for the function $\frac{1}{(1-x)^2}$ on the interval $(-1, 1)$. What is the interval of convergence of the resulting power series?

We have

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n \quad \text{for } -1 < x < 1.$$

The interval of convergence of the new series is $(-1, 1)$.

3. Substitute $x = -x^2$ into $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ and integrate term-by-term to obtain a series representation for $\arctan x$. What is the interval of convergence for the resulting power series?

Substituting $-x^2$ into the geometric series gives

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} \quad \text{for } -1 < x < 1$$

since $|x| < 1$ if and only if $|-x^2| < 1$. We now integrate term-by-term:

$$\arctan x + C = \int \frac{dx}{1+x^2} = \int \left(\sum_{n=0}^{\infty}(-1)^n x^{2n}\right) dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}.$$

Evaluating both sides at $x = 0$ gives $C = 0$. Hence

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } -1 < x < 1.$$

The new series has interval of convergence $[-1, 1)$.