Arclength, area, and slope in polar coordinates

Briefly, polar coordinates describe points in the Cartesian plane based on their distance from the origin and the angle they make with the positive x-axis:

$$r^2 = x^2 + y^2,\ \tan(\theta) = y/x \iff x = r\cos\theta,\ y = r\sin\theta.$$

Given a polar curve $r = r(\theta)$, $a \leq \theta \leq b$ we can calculate its length using the same method as for a parametric curve. We have

$$(x, y) = (r\cos\theta, r\sin\theta)$$

so that

$$\begin{aligned} L &= \int_a^b \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} d\theta = \int_a^b \sqrt{\left(\frac{d}{d\theta}(r\cos\theta)\right)^2 + \left(\frac{d}{d\theta}(r\sin\theta)\right)^2} d\theta = \dots \\ &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta. \end{aligned}$$

For the area between the origin and a polar curve, we need to know the area of a sector of a circle,

$$A_\theta = \frac{\theta}{2\pi}\pi r^2 = \frac{1}{2}r^2\theta,$$

as this is our basic unit of area (like a rectangle in Cartesian coordinates). We approximate the radius as constant on short angle intervals $[\theta_{i-1}, \theta_i]$, add up the area of the small sectors we obtain, and take a limit as the mesh of the partition goes to zero:

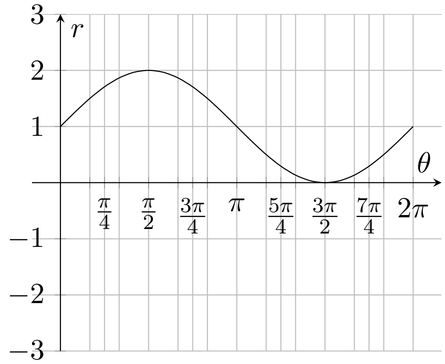
$$\int_a^b \frac{1}{2}r^2 d\theta = \lim_{n\to\infty}\sum_{i=1}^n \frac{1}{2}r(\theta_i^*)\Delta\theta_i.$$

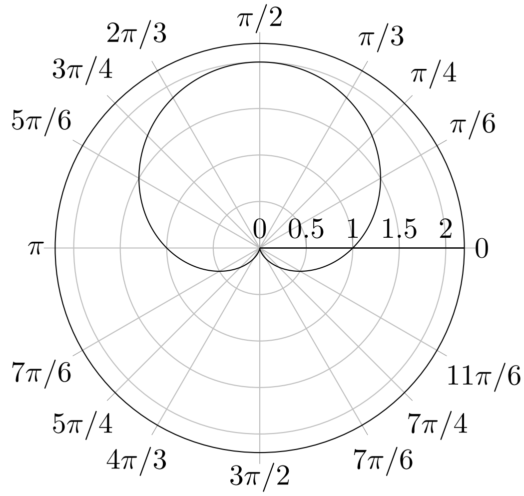
We can also talk about the slope dy/dx of polar curves, once again treating them as parametric curves $(x(\theta), y(\theta))$:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

In the problems below, the arclength integrals are not feasible (except for the cardioid), so just express the arclength as a definite integral.

Sketch $r = 1 + \sin\theta$ (cardioid), find the length of the curve, and find the area it encloses.





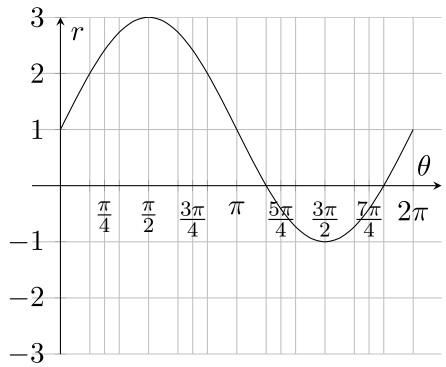
The length of the curve is

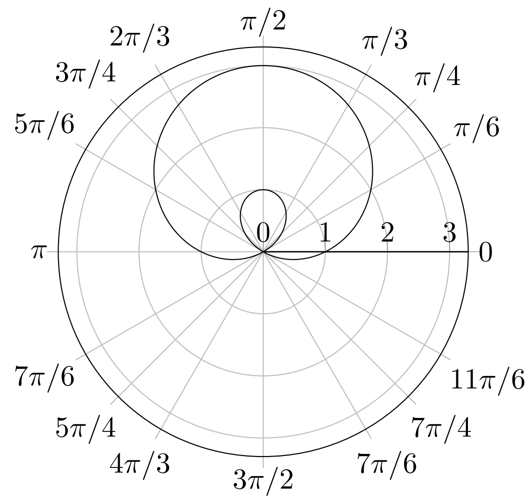
$$L = \int_0^{2\pi} ((1 + \sin\theta)^2 + (\cos\theta)^2)^{1/2} d\theta$$

and the area inside the curve is

$$A = \frac{1}{2}\int_0^{2\pi} (1 + \sin\theta)^2 d\theta.$$

Sketch $r = 1 + 2\sin\theta$ (limaçon), find the length of the curve, the area between the loops, and the tangent lines through the origin.





The length of the curve is

$$L = \int_0^{2\pi} \int ((1 + 2\sin\theta)^2 + (2\cos\theta)^2)^{1/2} d\theta.$$

The area inside the big loop but outside the small loop is given by

$$A = \frac{1}{2}\int_{-\pi/6}^{7\pi/6} (1 + 2\sin\theta)^2 d\theta - \frac{1}{2}\int_{7\pi/6}^{11\pi/6} (1 + 2\sin\theta)^2 d\theta,$$

the first integral is the area inside both, and the second is the area inside the small loop. Other choices for the domains on the integrals are possible.

The values of θ for which the curve goes through the origin (i.e. $r = 0$) are

$$0 = r = 1 + 2\sin\theta, \ \sin\theta = -1/2, \ \theta = 7\pi/6, \ 11\pi/6.$$

The slope of the tangent line at any value of θ is given by

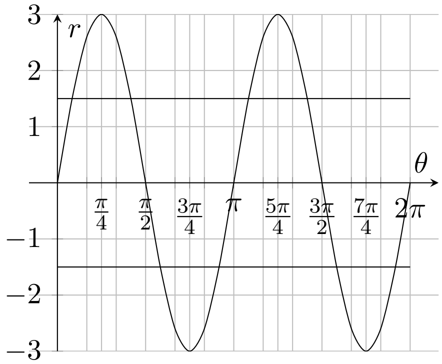
$$\frac{dy}{dx} = \frac{(2\cos\theta)\sin\theta + (1 + 2\sin\theta)\cos\theta}{(2\cos\theta)\cos\theta - (1 + 2\sin\theta)\sin\theta}$$

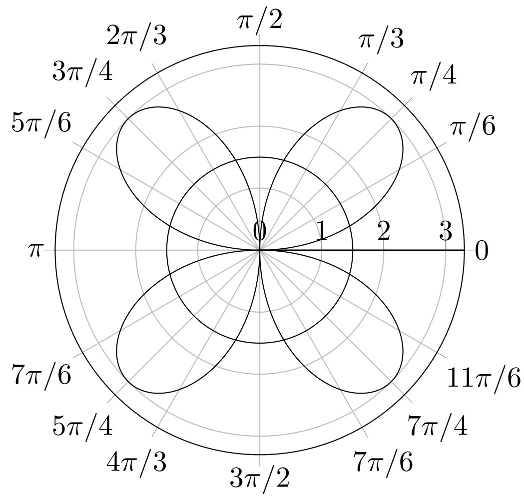
At $\theta = 7\pi/6, 11\pi/6$ these are

$$\left.\frac{dy}{dx}\right|_{\theta=7\pi/6} = 1/\sqrt{3}, \ \left.\frac{dy}{dx}\right|_{\theta=11\pi/6} = -1/\sqrt{3},$$

so the tangent lines through the origin are $y = \pm x/\sqrt{3}$.

Sketch $r = 3\sin(2\theta)$, find the length of the curve, the area it encloses, and the tangent line at $\theta = \pi/4$. Find the area inside the curve but outside the circle $r = 3/2$.





The length of the curve is

$$L = \int_0^{2\pi} \int ((3\sin(2\theta))^2 + (6\cos\theta)^2)^{1/2} d\theta$$

and the area inside the curve is

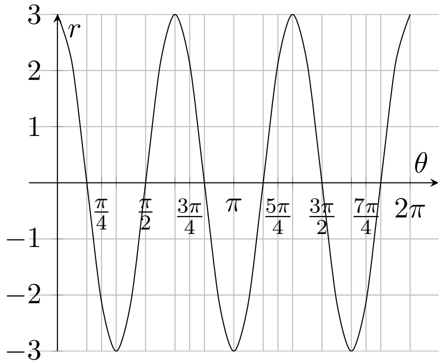
$$A = \frac{1}{2}\int_0^{2\pi} (3\sin(2\theta))^2 d\theta.$$

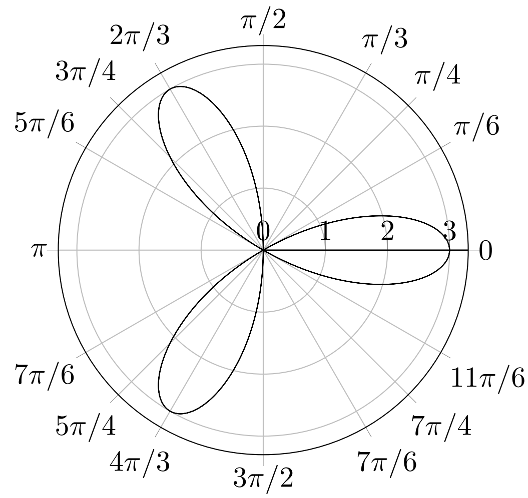
The slope of the tangent line at $\theta = \pi/4$ is given by

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = -1,$$

so the tangent line is $y - 3/\sqrt{2} = -(x - 3/\sqrt{2})$.

Sketch $r = 3\cos(3\theta)$, find the length of the curve, the area it encloses, and the tangent line at $\theta = 2\pi/3$.





Note that this curve is traced out once in an interval of length π. The area enclosed and arclength are given by

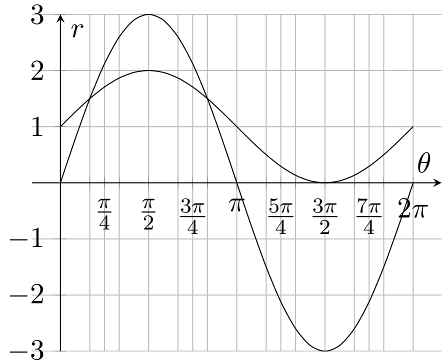
$$A = \frac{1}{2}\int_0^{\pi} (3\cos(3\theta))^2 d\theta,\ L = \int_0^{\pi} ((3\cos(3\theta))^2 + (-9\sin(3\theta))^2)^{1/2} d\theta.$$

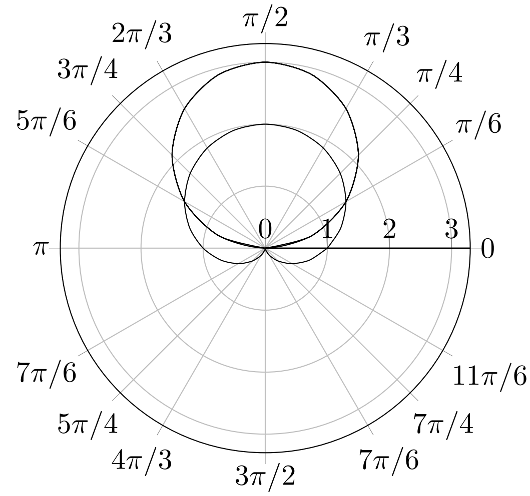
The slope of the tangent line at $\theta = 2\pi/3$ is

$$\left.\frac{dy}{dx}\right|_{\theta=2\pi/3} = \sqrt{3},$$

so the tangent line is $y - 3\sqrt{3}/2 = \sqrt{3}(x + 3/2)$.

Find the area inside the circle $r = 3\sin\theta$ but outside the cardioid $r = 1 + \sin\theta$.





The curves intersect where

$$3\sin\theta = r = 3/2,\ \sin\theta = 1/2,\ \theta = \pi/6, 5\pi/6.$$

The area inside the circle but outside the cardiod is

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6} (3/2)^2 d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1+\sin\theta)^2 d\theta.$$