Arclength, area, and slope in polar coordinates

Briefly, polar coordinates describe points in the Cartesian plane based on their distance from the origin and the angle they make with the positive x-axis:

$$r^2 = x^2 + y^2,\ \tan(\theta) = y/x \Longleftrightarrow\ x = r\cos\theta,\ y = r\sin\theta.$$

Given a polar curve $r = r(\theta)$, $a \le \theta \le b$ we can calculate its length using the same method as for a parametric curve. We have

$$(x, y) = (r\cos\theta, r\sin\theta)$$

so that

$$\begin{aligned} L &= \int_a^b \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}d\theta = \int_a^b \sqrt{\left(\frac{d}{d\theta}(r\cos\theta)\right)^2 + \left(\frac{d}{d\theta}(r\sin\theta)\right)^2} d\theta = \ldots \\ &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2}d\theta. \end{aligned}$$

For the area between the origin and a polar curve, we need to know the area of a sector of a circle,

$$A_\theta = \frac{\theta}{2\pi}\pi r^2 = \frac{1}{2}r^2\theta,$$

as this is our basic unit of area (like a rectangle in Cartesian coordinates). We approximate the radius as constant on short angle intervals $[\theta_{i-1}, \theta_i]$, add up the area of the small sectors we obtain, and take a limit as the mesh of the partition goes to zero:

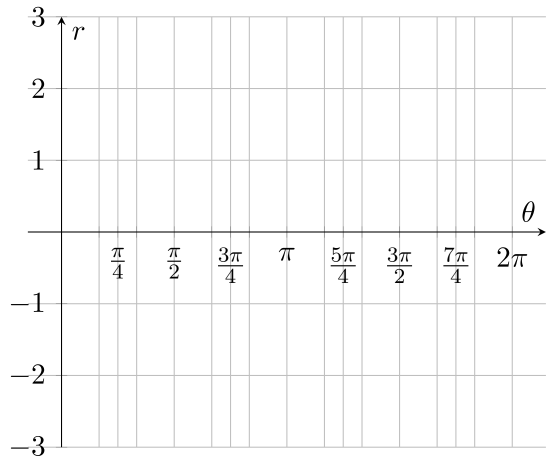
$$\int_a^b \frac{1}{2}r^2 d\theta = \lim_{n\to\infty}\sum_{i=1}^n \frac{1}{2}r(\theta_i^*)\Delta\theta_i.$$

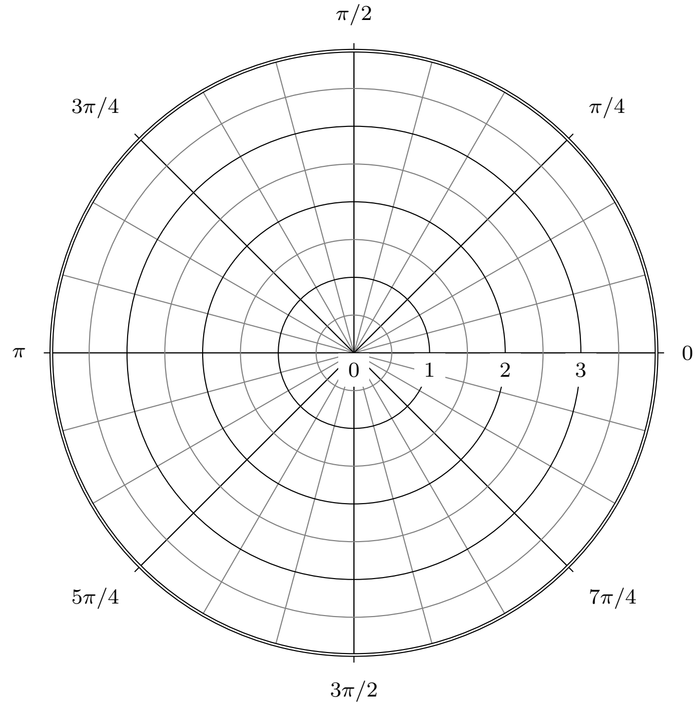
We can also talk about the slope dy/dx of polar curves, once again treating them as parametric curves $(x(\theta), y(\theta))$:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

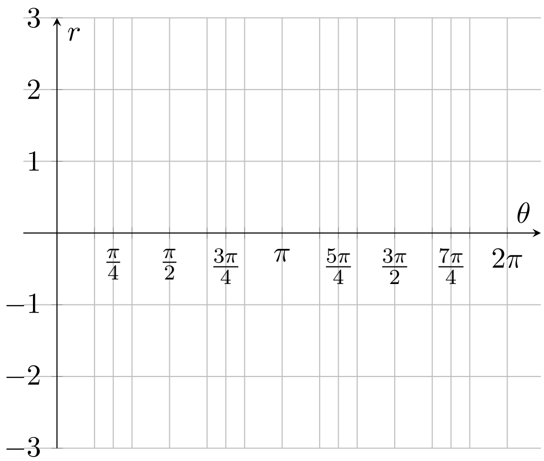
In the problems below, the arclength integrals are not feasible (except for the cardioid), so just express the arclength as a definite integral.

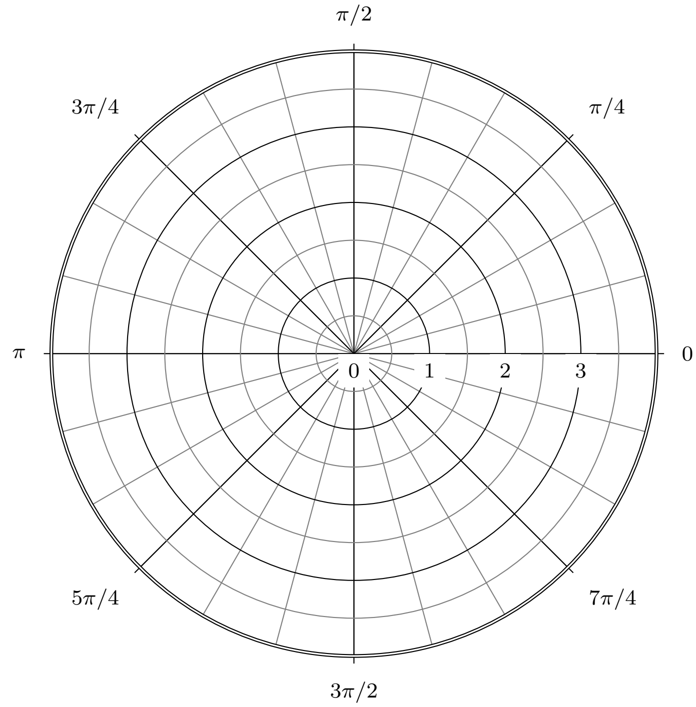
Sketch $r = 1 + \sin\theta$ (cardioid), find the length of the curve, and find the area it encloses.



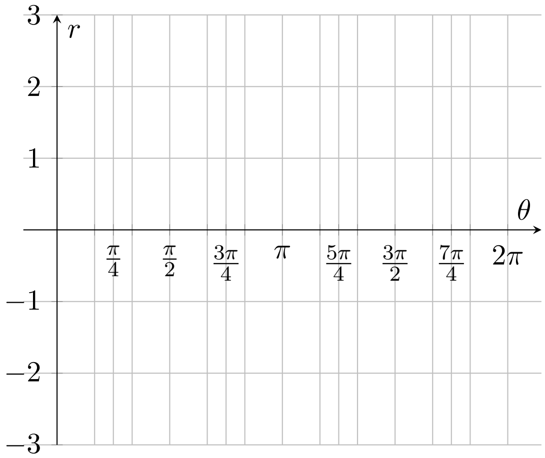


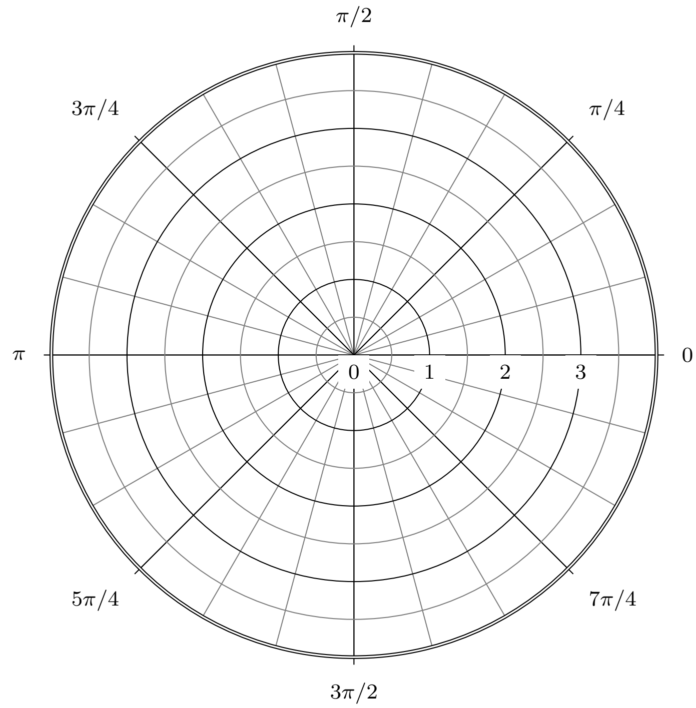
Sketch $r = 1 + 2\sin\theta$ (limaçon), find the length of the curve, the area between the loops, and the tangent lines through the origin.



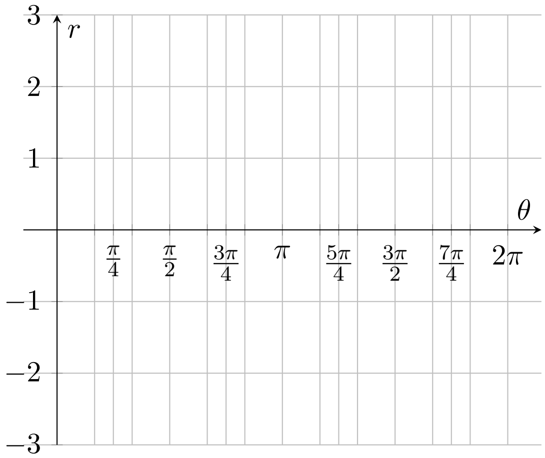


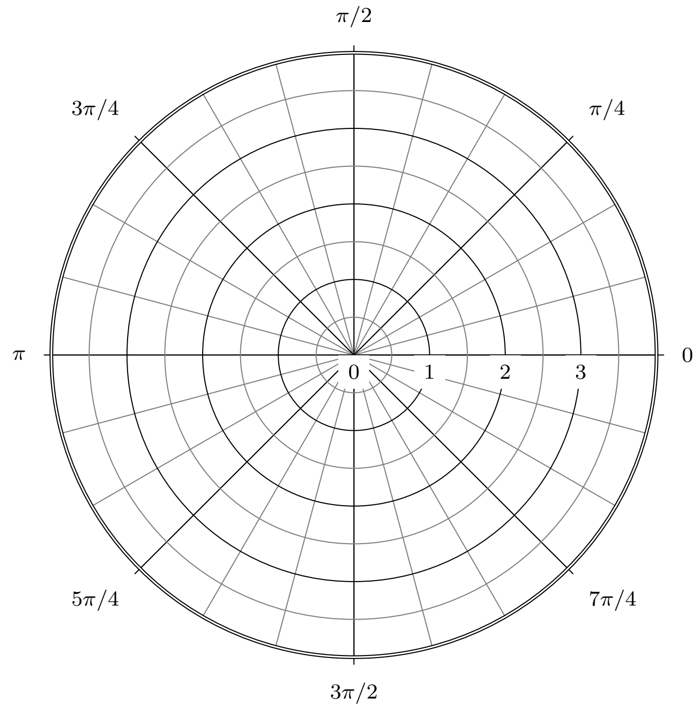
Sketch $r = 3\sin(2\theta)$, find the length of the curve, the area it encloses, and the tangent line at $\theta = \pi/4$. Find the area inside the curve but outside the circle $r = 3/2$.





Sketch $r = 3\cos(3\theta)$, find the length of the curve, the area it encloses, and the tangent line at $\theta = 2\pi/3$.





Find the area inside the circle $r = 3\sin\theta$ but outside the cardioid $r = 1 + \sin\theta$.