Basics

Here are some facts/tools for dealing with polynomials and rational functions (a quotient of polynomials).

- Every polynomial with real coefficients can be factored over the real numbers into a product of linear ($ax+b$) and irreducible quadratic (ax^2+bx+c, $b^2-4ac<0$) factors, e.g.
$$3x^5+2x^4+6x^3+4x^2+3x+2=(3x+2)(x^2+1)^2.$$

- You can divide two polynomials $\frac{a(x)}{b(x)}=q(x)+\frac{r(x)}{b(x)}$ to get a quotient $q(x)$ and remainder $r(x)$ with the degree of $r(x)$ less than the degree of $b(x)$, e.g.
$$\frac{2x^4+4x^3-x^2-10x+3}{x^3-3x+1}=2x+4+\frac{5x^2-1}{x^3-3x+1}.$$

- A quadratic polynomial ax^2+bx+c can be put in "standard form" (completing the square) as follows
$$ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}$$
e.g.
$$x^2+5x+3=(x+5/2)^2+3-\frac{25}{4}.$$

Partial fraction decomposition of rational functions

The idea of the partial fraction decomposition is to write a rational function as a sum of rational functions with "simpler" denominators, e.g.

$$\frac{5x^7+6x^6+13x^5+12x^4+9x^3+5x^2-5x-5}{x^6+2x^5+3x^4+4x^3+3x^2+2x+1}=5x-4+\frac{6x^5+4x^4+10x^3+7x^2-2x-1}{(x+1)^2(x^2+1)^2}$$
$$=5x-4+\frac{3}{x+1}+\frac{2x+1}{(x+1)^2}+\frac{x-2}{x^2+1}+\frac{2x-3}{(x^2+1)^2}.$$

The point, for us, is that the simpler pieces are easy/easier to integrate.

Here are a couple of examples.

- (two distinct linear factors). To decompose
$$\frac{5x-16}{x^2-7x+10}$$
we factor the denominator $x^2-7x+10=(x-2)(x-5)$ and try to solve the following for A and B
$$\frac{5x-16}{(x-2)(x-5)}=\frac{A}{x-2}+\frac{B}{(x-5)}.$$
Clearing denominators and combining like terms gives
$$5x-16=A(x-5)+B(x-2)=(A+B)x+(-5A-2B).$$

Equating coefficients of powers of x gives the system of linear equations

$$\begin{aligned} A + B &= 5 \\ -5A - 2B &= -16. \end{aligned}$$

There are various ways of solving linear systems (solve for one of the variables in the first equation and substitute into the second, add a multiple of one equation to the other to eliminate one of the variables, etc.). In any case we get

$$A = 2, \; B = 3,$$

so that

$$\frac{5x - 16}{x^2 - 7x + 10} = \frac{2}{x - 2} + \frac{3}{(x - 5)}.$$

- (repeated linear factor). To decompose

$$\frac{4x + 3}{x^2 + 2x + 1} = \frac{4x + 3}{(x + 1)^2},$$

we try solving for A, B, and C in the following (note the two terms, one for each power of the repeated linear factor)

$$\frac{4x + 3}{(x + 1)^2} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2}.$$

Clearing denominators and combining like terms gives

$$4x + 3 = A(x + 1) + B = Ax + (A + B).$$

Equating powers of x gives a system of linear equations

$$\begin{aligned} A &= 4 \\ A + B &= 3, \end{aligned}$$

which gives

$$A = 4, \; B = -1,$$

and

$$\frac{4x + 3}{x^2 + 2x + 1} = \frac{4}{x + 1} + \frac{-3}{(x + 1)^2}.$$

- (a linear and quadratic factor). To decompose

$$\frac{5x^2 + 3x + 1}{x^3 + 2x^2 + x + 2} = \frac{5x^2 + 3x + 1}{(x + 2)(x^2 + 1)}$$

we solve the following for A, B, and C (note the linear term $Bx + C$ for the irreducible quadratic factor)

$$\frac{5x^2 + 3x + 1}{(x + 2)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}.$$

Clearing denominators and combining like terms gives

$$5x^2 + 3x + 1 = A(x^2 + 1) + (Bx + C)(x + 2) = (A + B)x^2 + (2B + C)x + (A + 2C).$$

Equating coefficients of powers of x gives a linear system

$$\begin{aligned} A + B &= 5 \\ 2B + C &= 3 \\ A + 2C &= 1, \end{aligned}$$

with solution

$$A = 3,\ B = 2,\ C = -1.$$

Hence

$$\frac{5x^2 + 3x + 1}{x^3 + 2x^2 + x + 2} = \frac{3}{x+1} + \frac{2x - 1}{x^2 + 1}.$$

- (repeated quadratic factors). For a repeated quadratic factor, you will need linear denominators $A_i x + B_i$ for each power of the repeated factor. For example, to decompose

$$\frac{4x^5 + 3x^4 + 16x^3 + 12x^2 + 16x + 13}{(x^2+2)^3},$$

we need to solve

$$\frac{x^5 + x^4 + 7x^3 + 6x^2 + 10x + 9}{(x^2+2)^3} = \frac{A_1 x + B_1}{x^2 + 2} + \frac{A_2 x + B_2}{(x^2+2)^2} + \frac{A_3 x + B_3}{(x^2+2)^3},$$

which gives a linear system

$$\begin{aligned} A_1 &= 1 \\ B_1 &= 1 \\ 4A_1 + A_2 &= 7 \\ 4B_1 + B_2 &= 6 \\ 4A_1 + 2A_2 + A_3 &= 10 \\ 4B_1 + 2B_2 + B_1 &= 9, \end{aligned}$$

with solution

$$A_1 = 1,\ B_1 = 1,\ A_2 = 3,\ B_2 = 2,\ A_3 = 0,\ B_3 = 1.$$

Integrating the simple pieces

After completing the square and pulling out constants, you will only have to integrate the following to integrate ANY RATIONAL FUNCTION:

$$\int \frac{dx}{x+a} dx = \ln|x+a|,$$

$$\int \frac{dx}{(x+a)^n} = \frac{1}{(1-n)(x+a)^{n-1}},\ n > 1,$$

$$\int \frac{x}{(x^2+a^2)^n} dx = \frac{1}{2} \int \frac{du}{u^n}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan(x/a),$$

and finally (only with some repeated quadratic factors, so don't worry about memorizing this)

$$\int \frac{dx}{(x^2+a^2)^n} = \frac{2x}{4a^2(n-1)(x^2+a^2)^{n-1}} + \frac{2(2n-3)}{4a^2(n-1)} \int \frac{dx}{(x^2+a^2)^{n-1}}.$$

Problems

1. $\displaystyle\int \frac{x^2 - x + 5}{x^2 + x - 6} dx$

   After dividing and factoring the denominator, the integrand is

   $$\frac{x^2 - x + 5}{x^2 + x - 6} = 1 + \frac{-2x + 11}{x^2 + x - 6} = 1 + \frac{-2x + 11}{(x+3)(x-2)},$$

   so we want the partial fraction decomposition of $\frac{-2x+11}{(x+3)(x-2)}$, i.e. we want to solve

   $$\frac{-2x + 11}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}.$$

   Clearing denominators and combining like terms gives

   $$-2x + 11 = A(x-2) + B(x+3) = (A+B)x - 2A + 3B.$$

   Equating coefficients of powers of x gives a system of linear equations

   $$\begin{aligned} -2 &= A + B \\ 11 &= -2A + 3B, \end{aligned}$$

   with solution

   $$A = -17/5, \ B = 7/5.$$

   Hence the integral is

   $$\int dx - \frac{17}{5} \int \frac{dx}{x+3} + \frac{7}{5} \int \frac{dx}{x-2} = x - \frac{17}{5} \ln|x+3| + \frac{7}{5} \ln|x-2|.$$

2. $\displaystyle\int \frac{y^2 + 3}{y^3 - 3y^2 + 3y - 1} dy$

   The integrand is

   $$\frac{y^2 + 3}{y^3 - 3y^2 + 3y - 1} = \frac{y^2 + 3}{(y-1)^3} = \frac{A}{y-1} + \frac{B}{(y-1)^2} + \frac{C}{(y-1)^3}.$$

   Clearing denominators gives

   $$\begin{aligned} y^2 + 3 &= A(y-1)^2 + B(y-1) + C \\ &= Ay^2 + (B - 2A)y + A - B + C, \end{aligned}$$

   and we must solve the system of linear equations

   $$\begin{aligned} A &= 1 \\ B - 2A &= 0 \\ A - B + C &= 3, \end{aligned}$$

   to get

   $$A = 1, \ B = 2, \ C = 4.$$

   Hence the integral is

   $$\int \frac{dy}{y-1} + \int \frac{2dy}{(y-1)^2} + \int \frac{4dy}{(y-1)^3} = \ln|y-1| - 2(y-1)^{-1} - 2(y-1)^{-2}.$$

3. $\displaystyle\int \frac{dw}{w^2 - 2w + 10}$

Completing the square, the denominator is $(w-1)^2 + 9$ and the integral is

$$\frac{1}{3}\arctan\left(\frac{x-1}{3}\right).$$

4. $\displaystyle\int \frac{dx}{x^3+1}$

The integrand is

$$\frac{1}{1+x^3} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1},$$

which leads to

$$1 = (A+B)x^2 + (-A+B+C)x + (A+C),$$

with solution

$$A = \frac{1}{3},\ B = -\frac{1}{3},\ C = \frac{2}{3}.$$

Hence the integral becomes

$$\frac{1}{3}\int\left(\frac{1}{x+1} + \frac{-x+2}{x^2-x+1}\right)dx = \frac{1}{3}\ln|x+1| + \frac{1}{3}\int\frac{-x+2}{x^2-x+1}dx.$$

For the remaining integral, we complete the square and make a substitution to get

$$\begin{aligned}
\int \frac{-x+2}{(x-1/2)^2+3/4}dx &= \int \frac{-u+3/2}{u^2+3/4}du \\
&= -\int \frac{u}{u^2+3/4}du + \frac{3}{2}\int\frac{du}{u^2+3/4} \\
&= -\frac{1}{2}\int\frac{dv}{v} + \frac{3}{2}\frac{2}{\sqrt{3}}\arctan(2u/\sqrt{3}) \\
&= -\frac{1}{2}\ln|u^2+3/4| + \sqrt{3}\arctan((2x-1)/\sqrt{3}) \\
&= -\frac{1}{2}\ln|(x-1/2)^2+3/4| + \sqrt{3}\arctan((2x-1)/\sqrt{3}),
\end{aligned}$$

(where $u = x - 1/2$, $v = u^2 + 3/4$ above). Hence our final result is

$$\int\frac{dx}{x^3+1} = \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2-x+1| + \frac{1}{\sqrt{3}}\arctan((2x-1)/\sqrt{3}).$$

5. $\displaystyle\int \frac{-z^3 - 26z^2 - 28z - 120}{z^4 - 16} dz$

The integrand is

$$\frac{-z^3 - 26z^2 - 28z - 120}{z^4 - 16} = \frac{-z^3 - 26z^2 - 28z - 120}{(z^2+4)(z+2)(z-2)} = \frac{Az+B}{z^2+4} + \frac{C}{z+2} + \frac{D}{z-2}.$$

We must solve

$$\begin{aligned}&-z^3 - 26z^2 - 28z - 120 = (Az+B)(z+2)(z-2) + C(z-2)(z^2+4) + D(z+2)(z^2+4)\\ &= (A+C+D)z^3 + (B-2C+2D)z^2 + (-4A+4C+4D)z + (-4B-8C+8D).\end{aligned}$$

More compactly, we can reduce

$$\begin{pmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & -2 & 2 & -26 \\ -4 & 0 & 4 & 4 & -28 \\ 0 & -4 & -8 & 8 & -120 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -9 \end{pmatrix}$$

to get $A = 3$, $B = 2$, $C = 5$, $D = -9$. Hence the integral is

$$\begin{aligned}&\int \frac{3z+2}{z^2+4}dz + \int \frac{5dz}{z+2} + \int \frac{-9dz}{z-2} = \frac{3}{2}\int \frac{2z}{z^2+4}dz + 2\int \frac{dz}{z^2+4} + 5\int \frac{dz}{z+2} - 9\int \frac{dz}{z-2}\\ &= \frac{3}{2}\ln(z^2+4) + \arctan(z/2) + 5\ln|z+2| - 9\ln|z-2|.\end{aligned}$$