Basics

Here are some facts/tools for dealing with polynomials and rational functions (a quotient of polynomials).

- Every polynomial with real coefficients can be factored over the real numbers into a product of linear ($ax + b$) and irreducible quadratic ($ax^2 + bx + c$, $b^2 - 4ac < 0$) factors, e.g.
$$3x^5 + 2x^4 + 6x^3 + 4x^2 + 3x + 2 = (3x + 2)(x^2 + 1)^2.$$

- You can divide two polynomials $\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}$ to get a quotient $q(x)$ and remainder $r(x)$ with the degree of $r(x)$ less than the degree of $b(x)$, e.g.
$$\frac{2x^4 + 4x^3 - x^2 - 10x + 3}{x^3 - 3x + 1} = 2x + 4 + \frac{5x^2 - 1}{x^3 - 3x + 1}.$$

- A quadratic polynomial $ax^2 + bx + c$ can be put in "standard form" (completing the square) as follows
$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$
e.g.
$$x^2 + 5x + 3 = (x + 5/2)^2 + 3 - \frac{25}{4}.$$

Partial fraction decomposition of rational functions

The idea of the partial fraction decomposition is to write a rational function as a sum of rational functions with "simpler" denominators, e.g.

$$\frac{5x^7 + 6x^6 + 13x^5 + 12x^4 + 9x^3 + 5x^2 - 5x - 5}{x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1} = 5x - 4 + \frac{6x^5 + 4x^4 + 10x^3 + 7x^2 - 2x - 1}{(x + 1)^2(x^2 + 1)^2}$$
$$= 5x - 4 + \frac{3}{x + 1} + \frac{2x + 1}{(x + 1)^2} + \frac{x - 2}{x^2 + 1} + \frac{2x - 3}{(x^2 + 1)^2}.$$

The point, for us, is that the simpler pieces are easy/easier to integrate.

Here are a couple of examples.

- (two distinct linear factors). To decompose
$$\frac{5x - 16}{x^2 - 7x + 10}$$
we factor the denominator $x^2 - 7x + 10 = (x - 2)(x - 5)$ and try to solve the following for A and B
$$\frac{5x - 16}{(x - 2)(x - 5)} = \frac{A}{x - 2} + \frac{B}{(x - 5)}.$$
Clearing denominators and combining like terms gives
$$5x - 16 = A(x - 5) + B(x - 2) = (A + B)x + (-5A - 2B).$$

Equating coefficients of powers of x gives the system of linear equations

$$\begin{aligned} A + B &= 5 \\ -5A - 2B &= -16. \end{aligned}$$

There are various ways of solving linear systems (solve for one of the variables in the first equation and substitute into the second, add a multiple of one equation to the other to eliminate one of the variables, etc.). In any case we get

$$A = 2, \; B = 3,$$

so that

$$\frac{5x - 16}{x^2 - 7x + 10} = \frac{2}{x - 2} + \frac{3}{(x - 5)}.$$

- (repeated linear factor). To decompose

$$\frac{4x + 3}{x^2 + 2x + 1} = \frac{4x + 3}{(x + 1)^2},$$

we try solving for A, B, and C in the following (note the two terms, one for each power of the repeated linear factor)

$$\frac{4x + 3}{(x + 1)^2} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2}.$$

Clearing denominators and combining like terms gives

$$4x + 3 = A(x + 1) + B = Ax + (A + B).$$

Equating powers of x gives a system of linear equations

$$\begin{aligned} A &= 4 \\ A + B &= 3, \end{aligned}$$

which gives

$$A = 4, \; B = -1,$$

and

$$\frac{4x + 3}{x^2 + 2x + 1} = \frac{4}{x + 1} + \frac{-3}{(x + 1)^2}.$$

- (a linear and quadratic factor). To decompose

$$\frac{5x^2 + 3x + 1}{x^3 + 2x^2 + x + 2} = \frac{5x^2 + 3x + 1}{(x + 2)(x^2 + 1)}$$

we solve the following for A, B, and C (note the linear term $Bx + C$ for the irreducible quadratic factor)

$$\frac{5x^2 + 3x + 1}{(x + 2)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}.$$

Clearing denominators and combining like terms gives

$$5x^2 + 3x + 1 = A(x^2 + 1) + (Bx + C)(x + 2) = (A + B)x^2 + (2B + C)x + (A + 2C).$$

Equating coefficients of powers of x gives a linear system

$$\begin{aligned} A + B &= 5 \\ 2B + C &= 3 \\ A + 2C &= 1, \end{aligned}$$

with solution

$$A = 3,\ B = 2,\ C = -1.$$

Hence

$$\frac{5x^2 + 3x + 1}{x^3 + 2x^2 + x + 2} = \frac{3}{x+1} + \frac{2x - 1}{x^2 + 1}.$$

- (repeated quadratic factors). For a repeated quadratic factor, you will need linear denominators $A_i x + B_i$ for each power of the repeated factor. For example, to decompose

  $$\frac{4x^5 + 3x^4 + 16x^3 + 12x^2 + 16x + 13}{(x^2 + 2)^3},$$

  we need to solve

  $$\frac{x^5 + x^4 + 7x^3 + 6x^2 + 10x + 9}{(x^2 + 2)^3} = \frac{A_1 x + B_1}{x^2 + 2} + \frac{A_2 x + B_2}{(x^2 + 2)^2} + \frac{A_3 x + B_3}{(x^2 + 2)^3},$$

  which gives a linear system

  $$\begin{aligned} A_1 &= 1 \\ B_1 &= 1 \\ 4A_1 + A_2 &= 7 \\ 4B_1 + B_2 &= 6 \\ 4A_1 + 2A_2 + A_3 &= 10 \\ 4B_1 + 2B_2 + B_1 &= 9, \end{aligned}$$

  with solution

  $$A_1 = 1,\ B_1 = 1,\ A_2 = 3,\ B_2 = 2,\ A_3 = 0,\ B_3 = 1.$$

Integrating the simple pieces

After completing the square and pulling out constants, you will only have to integrate the following to integrate ANY RATIONAL FUNCTION:

$$\int \frac{dx}{x + a} dx = \ln |x + a|,$$

$$\int \frac{dx}{(x + a)^n} = \frac{1}{(1 - n)(x + a)^{n-1}},\ n > 1,$$

$$\int \frac{x}{(x^2 + a^2)^n} dx = \frac{1}{2} \int \frac{du}{u^n}$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan(x/a),$$

and finally (only with some repeated quadratic factors, so don't worry about memorizing this)

$$\int \frac{dx}{(x^2 + a^2)^n} = \frac{2x}{4a^2(n - 1)(x^2 + a^2)^{n-1}} + \frac{2(2n - 3)}{4a^2(n - 1)} \int \frac{dx}{(x^2 + a^2)^{n-1}}.$$

Problems

1. $\displaystyle\int \frac{x^2 - x + 5}{x^2 + x - 6} dx$

2. $\displaystyle\int \frac{y^2 + 3}{y^3 - 3y^2 + 3y - 1} dy$

3. $\displaystyle\int \frac{dw}{w^2 - 2w + 10}$

4. $\displaystyle\int \frac{dx}{x^3 + 1}$

5. $\displaystyle\int \frac{-z^3 - 26z^2 - 28z - 120}{z^4 - 16}dz$