

Integral test

Suppose $(a_n)_{n=1}^{\infty} = (f(n))_{n=1}^{\infty}$ where f is a positive, decreasing function on $[1, \infty)$ which is integrable on each interval $[1, T)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x) dx$ converges. [I.e. if one converges/diverges, so does the other.] Moreover,

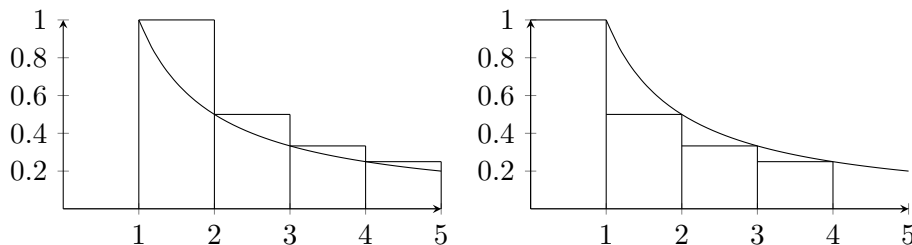
we have the following estimate on the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^N a_n$

$$\int_{N+1}^{\infty} f(x) dx \leq R_N \leq \int_N^{\infty} f(x) dx.$$

The series $\sum_{n=1}^{\infty} a_n$ can be interpreted as an improper integral of either of the piecewise constant functions

$$U(x) = a_n, \quad n < x \leq n+1, \quad L(x) = a_n, \quad n-1 < x \leq n,$$

whose graphs are above and below the graph of $f(x)$ respectively.



You can apply the integral test to *tails* of series, i.e. forget finitely many initial terms, so that the sequence of terms need only be *eventually* decreasing and *eventually* positive.

For example, consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. For $p \leq 0$ the terms do not approach zero, so the series diverges. For $p > 0$, we can compare the series to the improper integral

$$\int_1^{\infty} \frac{dx}{x^p}$$

since $f(x) = 1/x^p$ is decreasing, positive, and continuous on $[1, \infty)$. For $p \neq 1$ We have

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{T \rightarrow \infty} \int_1^T \frac{dx}{x^p} = \lim_{T \rightarrow \infty} \frac{1}{(1-p)x^{p-1}} \Big|_1^T = \lim_{T \rightarrow \infty} \frac{1}{(1-p)T^{p-1}} - \frac{1}{1-p} = \begin{cases} \frac{1}{p-1} & p > 1 \\ \infty & 0 < p < 1 \end{cases},$$

while for $p = 1$

$$\int_1^{\infty} \frac{dx}{x} = \lim_{T \rightarrow \infty} \ln T = \infty.$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}.$$

Use the integral test to show the convergence or divergence of the following series. Make sure the hypotheses are satisfied:

1.
$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

2.
$$\sum_{n=1}^{\infty} ne^{-n}$$

3.
$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$$

1. For the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, find N such that the remainder R_N is less than $1/1000$.

2. For the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$, estimate the remainder after adding the first 10 terms.