Integration by parts (product rule backwards)

The product rule states

$$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + f'(x)g(x).$$

Integrating both sides gives

$$f(x)g(x) = \int f(x)g'(x)dx + \int f'(x)g(x)dx.$$

Letting $f(x) = u$, $g(x) = v$, and rearranging, we obtain

$$\int u\ dv = uv - \int v\ du.$$

Some examples:

- $\int xe^x\ dx$. Differentiating $u = x$ gives a "simpler" function, $du = dx$, while integrating $dv = e^x dx$ gives $v = e^x$ which is no more difficult to integrate. Applying integration by parts gives
$$\int xe^x\ dx = xe^x - \int e^x\ dx = xe^x - e^x = e^x(x-1).$$
- $\int \ln x\ dx$. Differentiating $u = \ln x$ gives the "simpler" function $du = \frac{1}{x}dx$ while integrating $dv = dx$ gives $v = x$. Applying integration by parts, we obtain
$$\int \ln x\ dx = x\ln x - \int x\frac{1}{x}dx = x\ln x - x = x(\ln x - 1).$$
- $\int e^x \sin x\ dx$. We can integrate by parts twice and solve algebraically for the integral as follows:
$$\begin{aligned} I &= \int e^x \sin x\ dx = e^x \sin x - \int e^x \cos x\ dx = e^x \sin x - \left(e^x \cos x - \int e^x(-\sin x)dx\right) \\ &= e^x(\sin x - \cos x) - \int e^x \sin x\ dx = e^x(\sin x - \cos x) - I, \\ 2I &= e^x(\sin x - \cos x), \\ I &= \frac{e^x}{2}(\sin x - \cos x). \end{aligned}$$

Some for you to try:

1. $\displaystyle\int \frac{\ln y}{\sqrt{y}}dy$
   With

$$\begin{array}{cc} u = \ln y & dv = y^{-1/2}dy \\ du = dy/y & v = 2y^{1/2} \end{array}$$

   we have

$$\int \frac{\ln y}{\sqrt{y}}dy = 2y^{1/2}\ln y - 2\int y^{-1/2}dy = 2y^{1/2}\ln y - 4y^{1/2} = 2\sqrt{y}(\ln y - 2).$$

2. $\displaystyle\int \frac{x}{e^{2x}}dx$
   With

$$\begin{array}{cc} u = x & dv = e^{-2x}dx \\ du = dx & v = -e^{-2x}/2 \end{array}$$

   we have

$$\int \frac{x}{e^{2x}}dx = \int xe^{-2x}dx = -\frac{xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}dx = -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} = -\frac{e^{-2x}}{2}(x - 1/2).$$

3. $\displaystyle\int x \ln x \; dx$
   With

$$\begin{array}{cc} u = \ln x & dv = xdx \\ du = dx/x & v = x^2/2 \end{array}$$

   we have

$$\int x\ln x \; dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x \; dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} = \frac{x^2}{2}(\ln x - 1/2)$$

4. $\displaystyle\int_0^1 \arccos z \; dz$
   With

$$\begin{array}{cc} u = \arccos z & dv = dz \\ du = \frac{-dz}{\sqrt{1-z^2}} & v = z \end{array}$$

   we have

$$\int_0^1 \arccos z \; dz = z\arccos z\Big|_0^1 + \int_0^1 \frac{z}{\sqrt{1-z^2}}dz.$$

   We now make the substitution $u = 1 - z^2$, $du = -2zdz$ to obtain

$$\int_0^1 \frac{z}{\sqrt{1-z^2}}dz = -\frac{1}{2}\int_1^0 u^{-1/2}du = -\sqrt{u}\Big|_1^0 = 1.$$

   Hence

$$\int_0^1 \arccos z \; dz = z\arccos z\Big|_0^1 + \int_0^1 \frac{z}{\sqrt{1-z^2}} = 0 - 0 + 1 = 1.$$

5. $\displaystyle\int x^3\sqrt{1+x^2}\,dx$

You don't need integration by parts. With the substitution

$$u = 1+x^2,\ du = 2xdx,\ x^2 = u-1,$$

we have

$$\int x^3\sqrt{1+x^2}\,dx = \frac{1}{2}\int \frac{(u-1)du}{\sqrt{u}} = \frac{1}{2}\int (u^{1/2}-u^{-1/2})du = \frac{1}{3}u^{3/2} - u^{1/2} = (u/3-1)\sqrt{u}$$
$$= \frac{1}{3}(x^2-2)\sqrt{1+x^2}.$$

However, you could use integration parts with

$$\begin{array}{cc} u = x^2 & dv = \frac{x}{\sqrt{1+x^2}} \\ du = 2xdx & v = \sqrt{1+x^2} \end{array}$$

(using a substitution to find $\int dv$). We get (with $w = 1+x^2$, $dw = 2xdx$)

$$\int x^3\sqrt{1+x^2}\,dx = x^2\sqrt{1+x^2} - \int 2x\sqrt{1+x^2} = x^2\sqrt{1+x^2} - \int \sqrt{w}\,dw = x^2\sqrt{1+x^2} - \frac{2}{3}u^{3/2}$$
$$= x^2\sqrt{1+x^2} - \frac{2}{3}(1+x^2)^{3/2} = \frac{1}{3}(x^2-2)\sqrt{1+x^2}.$$

6. $\displaystyle\int_1^{\sqrt{3}} \arctan(1/x)dx$

With

$$\begin{array}{cc} u = \arctan(1/x) & dv = dx \\ du = \frac{-dx}{x^2+1} & v = x \end{array}$$

we get

$$\int_1^{\sqrt{3}} \arctan(1/x)dx = x\arctan(1/x)\Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x\,dx}{1+x^2}.$$

With $w = 1+x^2$, $dw = 2xdx$, the resulting integral is

$$\int_1^{\sqrt{3}} \frac{x\,dx}{1+x^2} = \frac{1}{2}\int_2^4 \frac{dw}{w} = \ln 4 - \ln 2 = \ln 2,$$

so that the end result is

$$\int_1^{\sqrt{3}} \arctan(1/x)dx = x\arctan(1/x)\Big|_1^{\sqrt{3}} + \ln 2 = \sqrt{3}\frac{\pi}{6} - \frac{\pi}{4} + \ln 2 = \frac{(2\sqrt{3}-3)\pi}{12} + \ln 2.$$

7. $\int (\ln x)^2 \; dx$
With

$$\begin{array}{cc} u = (\ln x)^2 & dv = dx \\ du = \frac{2\ln x}{x} dx & v = x \end{array}$$

we get

$$\int (\ln x)^2 \; dx = x(\ln x)^2 - 2\int \ln x \; dx = x(\ln x)^2 - (x\ln x - x) = x(\ln x)^2 - x\ln x + x.$$

8. $\int t^2 \sin(\pi t) dt$
We will use integration by parts twice to get rid of the t^2. With

$$\begin{array}{cc} u = t^2 & dv = \sin(\pi t) dt \\ du = 2t dt & v = -\cos(\pi t)/\pi \end{array}$$

we get

$$\int t^2 \sin(\pi t) dt = \frac{-t^2}{\pi} \cos(\pi t) + \frac{2}{\pi} \int t \cos(\pi t) dt.$$

Integration by parts on the resulting integral with

$$\begin{array}{cc} u = t & dv = \cos(\pi t) dt \\ du = dt & v = \sin(\pi t)/\pi \end{array}$$

gives

$$\int t\cos(\pi t) dt = \frac{t}{\pi}\sin(\pi t) - \frac{1}{\pi}\int \sin(\pi t) dt = \frac{t}{\pi}\sin(\pi t) + \frac{\cos(\pi t)}{\pi^2}.$$

Hence

$$\int t^2 \sin(\pi t) dt = \frac{-t^2}{\pi}\cos(\pi t) + \frac{2t}{\pi^2}\sin(\pi t) + \frac{2\cos(\pi t)}{\pi^3}.$$

9. $\int e^{-x} \cos(\pi x) dx$
This is similar to the third example (on the first page).

10. $\int \cos(\sqrt{x}) dx$ (Make a substitution first.)
With the substitution $y = \sqrt{x}$, $dy = \frac{dx}{2\sqrt{x}}$, the integral becomes

$$\int \cos(\sqrt{x}) dx = 2\int y\cos(y) dy.$$

Now use integration by parts with

$$\begin{array}{cc} u = y & dv = \cos y \; dy \\ du = dy & v = \sin y. \end{array}$$

11. $\int \sin(\ln w) dw$ (Make a substitution first.)
With the substitution $x = \ln w$, $dx = dw/w$, the integral becomes

$$\int \sin(\ln w) dw = \int e^x \sin x \; dx.$$

Now integrate as in the third example (on the first page).

12. $\int \sec^3 \theta \; d\theta$
With

$$\begin{aligned} u &= \sec\theta & dv &= \sec^2\theta d\theta \\ du &= \sec\theta\tan\theta & v &= \tan\theta \end{aligned}$$

the integral becomes

$$\int \sec^3 \theta \; d\theta = \sec\theta \tan\theta - \int \tan^2\theta \sec\theta \; d\theta.$$

Writing $\tan^2\theta = \sec^2\theta - 1$ gives

$$\int \tan^2\theta \sec\theta \; d\theta = \int (\sec^2\theta - 1)\sec\theta \; d\theta = \int \sec^3\theta \; d\theta - \int \sec\theta \; d\theta = \int \sec^3\theta \; d\theta - \ln|\sec\theta + \tan\theta.$$

Hence

$$\begin{aligned} I &= \int \sec^3\theta \; d\theta = \sec\theta\tan\theta - \int \sec^3\theta \; d\theta + \ln|\sec\theta + \tan\theta|, \\ 2I &= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|, \\ I &= \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|). \end{aligned}$$