Integration by parts (product rule backwards)

The product rule states

$$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + f'(x)g(x).$$

Integrating both sides gives

$$f(x)g(x) = \int f(x)g'(x)dx + \int f'(x)g(x)dx.$$

Letting $f(x) = u$, $g(x) = v$, and rearranging, we obtain

$$\int u\ dv = uv - \int v\ du.$$

Some examples:

- $\int xe^x\ dx$. Differentiating $u = x$ gives a "simpler" function, $du = dx$, while integrating $dv = e^x dx$ gives $v = e^x$ which is no more difficult to integrate. Applying integration by parts gives
$$\int xe^x\ dx = xe^x - \int e^x\ dx = xe^x - e^x = e^x(x-1).$$

- $\int \ln x\ dx$. Differentiating $u = \ln x$ gives the "simpler" function $du = \frac{1}{x}dx$ while integrating $dv = dx$ gives $v = x$. Applying integration by parts, we obtain
$$\int \ln x\ dx = x \ln x - \int x \frac{1}{x} dx = x \ln x - x = x(\ln x - 1).$$

- $\int e^x \sin x\ dx$. We can integrate by parts twice and solve algebraically for the integral as follows:
$$\begin{aligned} I &= \int e^x \sin x\ dx = e^x \sin x - \int e^x \cos x\ dx = e^x \sin x - \left(e^x \cos x - \int e^x(-\sin x)dx \right) \\ &= e^x(\sin x - \cos x) - \int e^x \sin x\ dx = e^x(\sin x - \cos x) - I, \\ 2I &= e^x(\sin x - \cos x), \\ I &= \frac{e^x}{2}(\sin x - \cos x). \end{aligned}$$

Some for you to try:

1. $\displaystyle\int \frac{\ln y}{\sqrt{y}} dy$

2. $\displaystyle\int \frac{x}{e^{2x}} dx$

3. $\displaystyle\int x \ln x \ dx$

4. $\displaystyle\int_0^1 \arccos z \ dz$

5. $\int x^3\sqrt{1+x^2}\ dx$

6. $\int_1^{\sqrt{3}} \arctan(1/x)dx$

7. $\int (\ln x)^2\ dx$

8. $\int t^2 \sin(\pi t)dt.$

9. $\int e^{-x} \cos(\pi x) dx$

10. $\int \cos(\sqrt{x}) dx$ (Make a substitution first.)

11. $\int \sin(\ln w) dw$ (Make a substitution first.)

12. $\int \sec^3 \theta \; d\theta$