Comparison tests

Direct comparison

Suppose $0 \leq a_n \leq b_n$ are sequences of positive terms. Then (taking limits of partial sums), we have

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n.$$

This implies that

$$\text{if } \sum_{n=1}^{\infty} b_n \text{ converges, then } \sum_{n=1}^{\infty} a_n \text{ converges,}$$

and

$$\text{if } \sum_{n=1}^{\infty} a_n \text{ diverges, then } \sum_{n=1}^{\infty} b_n \text{ diverges.}$$

Limit comparison

Suppose a_n, b_n are sequences of positive terms and that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ (i.e. the sequences grow at the same rate). Then

$$\sum_{n=1}^{\infty} b_n \text{ converges if and only if } \sum_{n=1}^{\infty} a_n \text{ converges,}$$

i.e. either both series converge or both series diverges.

Problems

1. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k + k^2}{3^k - k + 1}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n\sin(1/n)}{n^{3/2}+n+1}$

3. $\displaystyle\sum_{m=1}^{\infty} m^{1/m-1}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$