Trigonometric substitution

Expressions such as

$$\sqrt{a^2-x^2},\ \sqrt{a^2+x^2},\ \sqrt{x^2-a^2},\ a>0 \text{ constant},$$

may remind you of writing one side of a right triangle in terms of the other

$$a^2+b^2=c^2 \implies c=\sqrt{a^2+b^2},\ a=\sqrt{c^2-b^2}.$$

We can eliminate the $\sqrt{\ }$ with the Pythagorean identities

$$\sin^2\theta+\cos^2\theta=1,\ \tan^2\theta+1=\sec^2\theta,\ (1+\cot^2\theta=\csc^2\theta)$$

by parameterizing x with trigonometric function

$$x=a\sin\theta,\ x=a\tan\theta,\ x=a\sec\theta.$$

With these substitutions, we have (for appropriate ranges of θ)

$$\begin{aligned}
\sqrt{a^2-x^2}&=\sqrt{a^2-(a\sin\theta)^2}=a\sqrt{1-\sin^2\theta}=a\sqrt{\cos^2\theta}=a\cos\theta\\
\sqrt{a^2+x^2}&=\sqrt{a^2+(a\tan\theta)^2}=a\sqrt{1+\tan^2\theta}=a\sqrt{\sec^2\theta}=a\sec\theta\\
\sqrt{x^2-a^2}&=\sqrt{(a\sec\theta)^2-a^2}=a\sqrt{\sec^2\theta-1}=a\sqrt{\tan^2\theta}=a\tan\theta.
\end{aligned}$$

We can use the above to integrate some new functions. The answers you get can usually be rewritten in several ways, for instance

$$\arccos(x)=\arcsin(\sqrt{1-x^2})=\arctan(\sqrt{1/x^2-1}), \text{etc.}$$

We could also use the "hyperbolic" trigonometic functions (described in the next section), but they are not strictly part of the cirriculum.

- $\int\sqrt{1-x^2}dx$. With $x=\sin\theta$, $dx=\cos\theta\ d\theta$, we have

$$\begin{aligned}
\int\sqrt{1-x^2}dx&=\int\sqrt{1-\sin^2\theta}\cos\theta d\theta=\int\cos^2\theta d\theta=\frac{1}{2}\int(1+\cos(2\theta))d\theta\\
&=\frac{\theta}{2}+\frac{\sin(2\theta)}{4}=\frac{1}{2}(\theta+\sin\theta\cos\theta).
\end{aligned}$$

  Now, we need to get back the variable $x=\sin\theta$, $\theta=\arcsin x$. The right triangle with angle θ, hypothenuse 1, opposite side x and adjacent side $\sqrt{1-x^2}$ (draw a picture) satisfies $x=\sin\theta$ and tells us that $\cos\theta=\sqrt{1-x^2}$. Hence

$$\frac{1}{2}(\theta+\sin\theta\cos\theta)=\frac{1}{2}(\arcsin x+x\sqrt{1-x^2}).$$

- $\int\frac{dx}{\sqrt{x^2-4}}$. With $x=2\sec\theta$, $dx=2\sec\theta\tan\theta d\theta$, we have

$$\begin{aligned}
\int\frac{dx}{\sqrt{x^2-4}}&=\int\frac{2\sec\theta\tan\theta d\theta}{\sqrt{4\sec^2\theta-4}}=\int\frac{2\sec\theta\tan\theta d\theta}{2\sqrt{\sec^2\theta-1}}=\int\frac{\sec\theta\tan\theta d\theta}{\sqrt{\tan^2\theta}}\\
&=\int\sec\theta d\theta=\ln|\sec\theta+\tan\theta|.
\end{aligned}$$

We need to switch back to the variable $x = 2\sec\theta$, with $\theta = \operatorname{arcsec}(x/2)$. The right triangle with angle θ, hypothenuse x, adjacent side 2, and opposite side $\sqrt{x^2-4}$ (draw a picture) satisfies $2\sec\theta = x$ and shows that $\tan\theta = \frac{1}{2}\sqrt{x^2-4}$. Hence

$$\ln|\sec\theta + \tan\theta| = \ln\left|\frac{x}{2} + \frac{1}{2}\sqrt{x^2-4}\right|.$$

- Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The top half of the ellipse is given by the graph of $y = b\sqrt{1-(x/a)^2}$. The area of one quarter of the ellipse is given by

$$\int_0^a b\sqrt{1-(x/a)^2}dx.$$

With $x = a\sin\theta$, $dx = a\cos\theta d\theta$, we have

$$\int_0^a b\sqrt{1-(x/a)^2}dx = \int_0^{\pi/2} b\sqrt{1-\sin^2\theta}a\cos\theta d\theta = ab\int_0^{\pi/2}\cos^2\theta$$
$$= \frac{ab}{2}\int_0^{\pi/2}(1+\cos(2\theta))d\theta = \frac{ab}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right)\Big|_0^{\pi/2} = \frac{\pi ab}{4}.$$

Hence the total area of the ellipse is πab. When $a = b = r$ (i.e. the ellipse is a circle of radius r), we get πr^2 as expected.

- $\displaystyle\int \frac{dx}{x\sqrt{x^2-9}}$. With $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta d\theta$, we have

$$\int \frac{dx}{x\sqrt{x^2-9}} = \int \frac{3\sec\theta\tan\theta d\theta}{3\sec\theta\sqrt{9\sec^2\theta-9}} = \frac{1}{3}\int \frac{\sec\theta\tan\theta d\theta}{\sec\theta\sqrt{\sec^2\theta-1}} = \frac{1}{3}\int \frac{\sec\theta\tan\theta d\theta}{\sec\theta\sqrt{tan^2\theta}}$$
$$= \frac{1}{3}\int d\theta = \frac{\theta}{3}.$$

Switching to x we get

$$\frac{\theta}{3} = \frac{1}{3}\operatorname{arcsec}(x/3).$$

- $\displaystyle\int \frac{dx}{\sqrt{x^2+4x+20}}$. We complete the square and substitute to get

$$\int \frac{dx}{\sqrt{x^2+4x+20}} = \int \frac{dx}{\sqrt{(x+2)^2+16}} = \int \frac{du}{\sqrt{u^2+4^2}}.$$

With $u = 4\tan\theta$, $du = 4\sec^2\theta d\theta$, we have

$$\int \frac{du}{\sqrt{u^2+4^2}} = \int \frac{4\sec^2\theta d\theta}{\sqrt{16\tan^2\theta+16}} = \int \frac{\sec^2\theta d\theta}{\sqrt{\tan^2\theta+1}} = \int \frac{\sec^2\theta d\theta}{\sqrt{\sec^2\theta}}$$
$$= \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta|.$$

Switching back to $u = 4\tan\theta$ (triangles blahblahblah) and then to $x = u-2$ we get

$$\ln|\sec\theta + \tan\theta| = \ln|\sqrt{1+(u/4)^2} + u/4| = \ln|\sqrt{1+((x+2)/4)^2} + (x+2)/4|.$$

- $\int \sqrt{1+x^2}dx$. With $x = \tan\theta$, $dx = \sec^2\theta \; d\theta$, we have

$$\int \sqrt{1+x^2}dx = \int \sqrt{1+\tan^2\theta}\sec^2\theta d\theta = \int \sec^3\theta d\theta$$
$$= \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|).$$

(The integral $\int \sec^3\theta\theta$ is on the quiz 1 take-home solutions). We need to get back to the variable $x = \tan\theta$, $\theta = \arctan x$. The right triangle with angle θ, hypothenuse $\sqrt{1+x^2}$, adjacent side 1, and opposite side x (draw a picture) satisfies $x = \tan\theta$ and shows that $\sec\theta = \sqrt{1+x^2}$. Hence

$$\frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) = \frac{1}{2}(x\sqrt{1+x^2} + \ln|x + \sqrt{1+x^2}|).$$

- $\int \sqrt{x^2-1}dx$. With $x = \sec\theta$, $dx = \sec\theta\tan\theta \; d\theta$, we have

$$\int \sqrt{x^2-1}dx = \int \sqrt{\sec^2\theta - 1}\sec\theta\tan\theta d\theta = \int \sec\theta\tan^2\theta d\theta = \int \sec\theta(1-\sec^2\theta)d\theta$$
$$= \int \sec\theta d\theta - \int \sec^3\theta d\theta = \ln|\sec\theta + \tan\theta| - \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|).$$

We need to get back to the variable $x = \sec\theta$, $\theta = \operatorname{arcsec} x$. The right triangle with angle θ, hypothenuse x, adjacent side 1, and opposite side $\sqrt{x^2-1}$ (draw a picture) satisfies $x = \sec\theta$ and shows that $\tan\theta = \sqrt{x^2-1}$. Hence

$$\ln|\sec\theta+\tan\theta| - \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|)$$
$$= \ln|x+\sqrt{x^2-1}| - \frac{1}{2}(x\sqrt{x^2-1} + \ln|x+\sqrt{x^2-1}|)$$
$$= \frac{1}{2}(\ln|x+\sqrt{x^2-1}| - x\sqrt{x^2-1}).$$

Extra-cirricular: hyperbolic trigonometric functions

The parameterization $x = \cos t$, $y = \sin t$ nicely discribes the unit circle, $x^2 + y^2 = 1$; as t increases, $(x(t), y(t))$ moves counter-clockwise around the circle at a constant speed.

There is a similar parameterization of the unit hyperbola $x^2 - y^2 = 1$ by the "hyperbolic" (as opposed to "circular") trigonometric functions, definied as

$$\cosh t = \frac{e^t + e^{-t}}{2}, \; \sinh t = \frac{e^t - e^{-t}}{2}.$$

One can verify that

$$\cosh^2 t - \sinh^2 t = 1, \; \frac{d}{dt}\cosh t = \sinh t, \; \frac{d}{dt}\sinh t = \cosh t,$$

(see page 227 of your text). So it seems natural to use these to integrate functions involving $\sqrt{x^2-a^2}$ and $\sqrt{a^2+x^2}$ (or even $\sqrt{a^2-x^2}$ if we use $\tanh t$ and $\operatorname{sech} t$).

The hyperbolic trigonometric functions are closely related to the circular trigonometric functions, in fact we have

$$\sin t = \frac{e^{it} - e^{-it}}{2i} = -i\sinh(it), \ \cos t = \frac{e^{it} + e^{-it}}{2} = \cosh(it), \ i = \sqrt{-1},$$

which we won't discuss, but could prove after we discuss power series later in the course. Another similarity comes from the fact that, $\sin t$, $\cos t$ are the solutions to the differential equations $y'' + y = 0$ with initial conditions $y(0) = 0$, $y'(0) = 1$ and $y(0) = 1$, $y'(0) = 0$ respectively, where as $\sinh t$ and $\cosh t$ are solutions of $y'' - y = 0$ with the same initial conditions. We will discuss some differential equation later in the course.