Separable differential equations

Given a first order equation of the form

$$f(y)\frac{dy}{dx} = g(x)$$

we can integrate both sides with respect to x (using a substitution on the left) to obtain

$$F(y) = G(x) + C$$

where $F' = f$, $G' = g$, and C is an arbitrary constant. If we can invert F, we can solve for y:

$$y = F^{-1}(G(x) + C).$$

Moreover, if we have an initial condition, $y(x_0) = y_0$, this determines a value for the constant C:

$$y(x_0) = y_0 = F^{-1}(G(x_0) + C),\ \ C = F(y_0) - G(x_0).$$

In general, a first order differential equation is separable if you can express it as

$$a(x)dx = b(y)dy,$$

and it can be solved by integrating both sides and then solving for y. DON'T FORGET THE CONSTANT OF INTEGRATION!

Solve the following initial value problems. [Solutions at the end.]

1. $\dfrac{dy}{dx} = -2y,\ \ y(0) = 1$

2. $u\dfrac{du}{dx} = 1,\ \ u(0) = 1$

3. $\dfrac{dx}{dt} + x = 1,\ \ x(0) = 0.1$

4. $\dfrac{du}{dt} = u + ut^2, \ u(0) = 5$

5. $\dfrac{dA}{dx} = xe^A, \ A(0) = 0$

6. $\dfrac{ds}{d\theta} = -s^2 \tan\theta, \ s(0) = 2$

7. $x(x+1)\dfrac{dy}{dx} = y^2, \ y(1) = 1$

8. $\dfrac{dP}{dx} = \dfrac{5P}{x}, \ P(1) = 3$

9. $\dfrac{dz}{dx} = xz^2 \sin(x^2), \ z(0) = 1$

Solutions

1. $y(x) = e^{-2x}$
2. $u(x) = \sqrt{2x+1}$
3. $x(t) = 1 - 0.9e^{-1}$
4. $u(t) = 5e^{t^3/3+t}$
5. $A(x) = -\ln(1 - x^2/2)$
6. $s(\theta) = \dfrac{2}{1 - \ln|\cos\theta|}$
7. $y(x) = \dfrac{1}{\ln|1 + 1/x| - \ln 2}$
8. $P(x) = 3x^5$
9. $z(x) = \dfrac{2}{1 + \cos(x^2)}$