MATH 2300-015 QUIZ 9 Due Tuesday, October 31st Name: ______________________

1. Starting with the geometric series, $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$, show that

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

[Hint: Evaluate $\arctan(x)$ at $1/\sqrt{3}$.]

We have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

for $|x| < 1$. Integrating term-by-term gives

$$\arctan x = \int \frac{dx}{1+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

for $|x| < 1$, after noting that the two functions agree at $x = 0$. Evaluating at $x = 1/\sqrt{3}$, we get

$$\pi/6 = \arctan(1/\sqrt{3}) = \sum_{n=0}^{\infty} \frac{(-1)^n(1/\sqrt{3})^{2n+1}}{2n+1} = \frac{1}{\sqrt{3}}\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)},$$

which gives the desired result after multiplying both sides by 6.

2. What is the radius of convergence of $\sum_{n=0}^{\infty} \dfrac{n^n}{n!}x^n$?

The ratio of successive coefficients is

$$\frac{(n+1)^{n+1}}{(n+1)!}\frac{n!}{n^n} = \left(\frac{n+1}{n}\right)^n = (1+1/n)^n \to e = 1/R,$$

so the radius of convergence is $R = 1/e$.

3. This problem will use power series to solve the initial value problem

$$y' = y,\ y(0) = 1.$$

(a) Suppose $y(x) = \sum_{n=0}^{\infty} c_n x^n$ is a solution. Differentiate y term-by-term and equate the coefficients of x^n on both sides of $y' = y$ to determine c_n.

We are supposing that

$$y = \sum_{n=0}^{\infty} c_n x^n = y' = \sum_{n=0}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n,\ c_0 = 1.$$

Equating the coefficients of x^n gives

$$c_n = (n+1)c_{n+1},\ c_{n+1} = \frac{c_n}{n+1},\ c_0 = 1,$$

so that $c_n = 1/n!$.

(b) Show that the resulting series converges for all values of x and differentiate term-by-term to verify that $y' = y$.

The resulting series is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

which has infinite radius of convergence

$$\frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0 = 1/R, \;\; R = \infty.$$

Differentiating term-by-term gives

$$y' = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = y,$$

verifying that this is in fact a solution to the differential equation above.