MATH 2300-015 QUIZ 8 Due Monday, October 23rd Name: ____________________

1. Determine whether the following series converge or diverge. If a series converges and the terms are not eventually positive, determine whether or not the convergence is absolute

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

Converges by direct comparison to $\displaystyle\sum_n \frac{1}{n^2}$. We have

$$0 \leq \frac{n}{n^3+1} = \frac{1}{n^2+1/n} \leq \frac{1}{n^2}$$

and $\displaystyle\sum_n \frac{1}{n^2}$ is a convergent p-series, $p = 2 > 1$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

Diverges by limit comparison to $\sum_n \frac{1}{n}$. We have positive terms $1/n$, $\frac{n^2+1}{n^3+1}$ and

$$\lim_{n\to\infty} \frac{1/n}{(n^2+1)/(n^3+1)} = \lim_{n\to\infty} \frac{n^3+1}{n^3+n} = \lim_{n\to\infty} \frac{1+1/n^3}{1+1/n^2} = 1.$$

Therefore the sum in question diverges because $\sum_n 1/n$ diverges (harmonic series, p-series, $p = 1 \leq 1$).

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5^n}$

Converges by the ratio test ($|a_{n+1}/a_n| \to 1/5$). We have

$$\lim_{n\to\infty} \frac{(n+1)^3/5^{n+1}}{n^3/5^n} = \lim_{n\to\infty} \frac{(1+1/n)^3}{5} = 1/5 < 1.$$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

Converges conditionally by the alternating series test. First note that the series does not converge absolutely

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n+1}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}},$$

a divergent p-series ($p = 1/2 < 1$).

However, the series is of the form $\sum_n (-1)^n b_n$ with $b_n = 1/\sqrt{n+1}$ and

$$0 < b_n, \ \lim_{n\to\infty} b_n = 0, \ \sqrt{n+1} > \sqrt{n} \Rightarrow \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}, \quad \text{i.e. } b_n \geq b_{n+1}.$$

Hence the series converges by the alternating series test.

(e) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Diverges by the integral test. The function $f(x) = \dfrac{1}{x(\ln x)^{1/2}}$ is positive and continuous on $(1, \infty)$, and is decreasing on $(1, \infty)$ since both x and $\sqrt{\ln x}$ are both increasing there. We have

$$\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim_{T\to\infty} \int_2^{T} \frac{dx}{x\sqrt{\ln x}} = \lim_{T\to\infty} \int_{\ln 2}^{\ln T} \frac{du}{\sqrt{u}} = 2\sqrt{u}\Big|_{\ln 2}^{\ln T} = \infty,$$

so that the series also diverges.

(f) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$

Diverges by the divergence test ($a_n \to \ln(1/3)$). We have

$$\lim_{n\to\infty} \ln\left(\frac{n}{3n+1}\right) = \ln\left(\lim_{n\to\infty} \frac{n}{3n+1}\right) = \ln\left(\lim_{n\to\infty} \frac{1}{3+1/n}\right) = \ln(1/3) \neq 0.$$

(g) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1}$

Converges conditionally by the alternating series test. Note that the series does not converge absolutely, say by limit comparison to $\sum_n 1\sqrt{n}$:

$$\frac{\sqrt{n}}{n+1}/(1/\sqrt{n}) = \frac{n}{n+1} \to 1 \text{ as } n \to \infty,$$

and $\sum_n 1/\sqrt{n}$ diverges (p-series, $p = 1/2 < 1$). However, the series is alternating, $\sum_n (-1)^n b_n$ with $b_n = \frac{\sqrt{n}}{n+1}$:

$$b_n > 0, \ \lim_{n\to\infty} b_n = 0,$$

and

$$\frac{\sqrt{n}}{n+1} \geq \frac{\sqrt{n+1}}{n+2} \iff \left(\frac{n+2}{n+1}\right)^{3/2} = \left(1 + \frac{1}{n+1}\right)^{3/2} \geq 1,$$

so that b_n is decreasing. Hence the series converges by the alternating series test.

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(3n)}{1+(1.2)^n}$

$\sum_n |a_n|$ converges by direct comparison to $\sum_n (1/1.2)^n$, a convergent geometric series. We have

$$0 \leq \left|\frac{\cos(3n)}{1+(1.2)^n}\right| \leq \frac{1}{(1.2)^n}$$

and $\sum_n (1/1.2)^n$ converges (common ratio $r = 1/1.2$ satisfies $|r| < 1$).

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{5^n n!}$

Conveges by the ratio test ($|a_{n+1}/a_n| \to 2/5$). We have

$$\left|\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2(n+1)-1)}{5^{n+1}(n+1)!} \frac{5^n n!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}\right| = \frac{2n+1}{5(n+1)} \to \frac{2}{5} < 1.$$

(j) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1+n}{3n}\right)^n$

Converges by limit comparison to $\sum_n (1/3)^n$. We have

$$\lim_{n\to\infty}\frac{(1/3)^n}{(\frac{1+n}{3n})^n}=\lim_{n\to\infty}\left(\frac{3n}{3n+3}\right)^n=\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^n=1/e,$$

so the series in question converges by limit comparison.

(k) $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{(2n+1)!}$

Converges by the ratio test ($|a_{n+1}/a_n|\to 0$). We have

$$\frac{(n+1)^{n+1}}{(2(n+1)+1)!}\frac{(2n+1)!}{n^n}=\frac{n+1}{(2n+3)(2n+2)}(1+1/n)^n\to 0\cdot e=0<1.$$

(l) $\displaystyle\sum_{n=1}^{\infty}\frac{8^n}{n!}$

Converges by the ratio test ($|a_{n+1}/a_n|\to 0$). We have

$$\frac{8^{n+1}}{(n+1)!}\frac{n!}{8^n}=\frac{8}{n+1}\to 0<1.$$

(m) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^n}{n^2}$

Diverges by the divergence test, $2^n/n^2\to\infty$.

(n) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$

Converges conditionally by the alternating series test ($\cos(n\pi)=(-1)^n$). We have

$$\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n},$$

which converges by the alternating series test ($0<1/n$ decreases to zero). The convergence is conditional since $\sum_n 1/n$ diverges (harmonic series, p-series, $p=1\le 1$).

(o) $\displaystyle\sum_{n=1}^{\infty}\frac{\tan(1/n)}{n^{3/2}}$

Converges by direct comparison to $(\pi/4)\sum_n n^{-3/2}$ or by limit comparison to $\sum_n 1/n^{5/2}$ ($\tan(x)\approx x$ for x small). Details omitted.

(p) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{2+\sin n}$

Diverges by the divergence test, since $1\le 2+\sin(n)\le 3$.

(q) $\displaystyle\sum_{n=1}^{\infty} \sin(1/n^2)$

Converges by limit comparison to $\sum_n 1/n^2$. We have positive terms $\sin(1/n^2)$, $1/n^2$ and

$$\lim_{n\to\infty} \frac{\sin(1/n^2)}{1/n^2} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$

The series $\sum_n 1/n^2$ is a convergent p-series ($p = 2 > 1$), so the series in question converges as well.

(r) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos(1/n^2)$

Diverges by the divergence test. We have

$$\lim_{n\to\infty} \cos(1/n^2) = \cos(\lim_{n\to\infty} 1/n^2) = \cos(0) = 1.$$

(s) $\displaystyle\sum_{n=1}^{\infty} 2^{-\ln n}$

Diverges. This is a p-series in disguise, $2^{-\ln n} = 1/n^{\ln 2}$, $p = \ln 2 < 1$.

(t) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

Converges by the integral test. The function $f(x) = xe^{-x^2}$ is positive and continuous on $(0, \infty)$ and decreasing on $[1, \infty)$ because

$$f'(x) = x(-2xe^{-x^2}) + e^{-x^2} = e^{-x^2}(1 - 2x^2) < 0 \text{ for } x > 1/\sqrt{2}.$$

By the integral test, the series converges if and only if the improper integral $\int_1^\infty f(x)dx$ converges. We have

$$\int_1^\infty xe^{-x^2}dx = \lim_{T\to\infty} \int_1^T xe^{-x^2}dx = \lim_{T\to\infty} \frac{1}{2} \int_1^{T^2} e^{-u}du = \frac{1}{2} \lim_{T\to\infty} \left(e^{-1} - e^{-T^2}\right) = \frac{1}{2e},$$

and the series converges.

2. Find the values of the following series telescoping or geometric series.

(a) $\displaystyle\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$

The sum is $-\ln 2$. We have

$$\sum_{n=2}^{N} \ln(1 - 1/n^2) = \sum_{n=2}^{N} \ln\left(\frac{(n-1)(n+1)}{n^2}\right) = \sum_{n=2}^{N} [\ln(n+1) + \ln(n-1) - 2\ln n]$$
$$= [\ln 3 + \ln 2 - 2\ln 2] + [\ln 4 + \ln 2 - 2\ln 3] + [\ln 5 + \ln 3 - 2\ln 4] +$$
$$\ldots + [\ln N + \ln(N-2) - 2\ln(N-1)] + [\ln(N+1) + \ln(N-1) - 2\ln N]$$
$$= -\ln 2 + \ln(N+1) - \ln N = -\ln 2 + \ln(1 + 1/N) \to -\ln 2.$$

(b) $\displaystyle\sum_{n=4}^{\infty} \frac{5 \cdot 2^{n-3} + 2 \cdot 3^{n-5}}{3 \cdot 5^{n-2}}$

The sum is 11/45 (the sum of two geometric series). We have

$$\sum_{n=4}^{\infty} \frac{5 \cdot 2^{n-3} + 2 \cdot 3^{n-5}}{3 \cdot 5^{n-2}} = \sum_{n=4}^{\infty} \frac{5 \cdot 2^{n-3}}{3 \cdot 5^{n-2}} + \sum_{n=4}^{\infty} \frac{2 \cdot 3^{n-5}}{3 \cdot 5^{n-2}} = \frac{2/15}{1-2/5} + \frac{2/225}{1-3/5} = 11/45,$$

recalling that the sum of a geometric series $\sum_{n=0}^{\infty} ar^n$ is given by $\frac{a}{1-r}$ where a is the first term of the series and r is the common ratio.

3. Use the integral or alternating series test remainder estimate for the following problems, first showing that the series in question actually converges.

(a) How many terms N of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ can we use to guarantee the remainder

$$R_N = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} - \sum_{n=2}^{N} \frac{1}{n(\ln n)^2}$$

is less than 0.1?

The series converges by the integral test, $\frac{1}{n(\ln n)^2} = f(n)$ where $f(x) = \frac{1}{x(\ln x)^2}$. The function is positive and decreasing for $x > 1$ so the series converges if and only if the associated improper integral converges

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\infty} \frac{du}{u^2} = \ln 2.$$

The remainder R_N satisfies

$$\int_{N+1}^{\infty} f(x)dx \le R_N \le \int_N^{\infty} f(x)dx$$

so the remainder R_N will be less than 0.1 if

$$\int_N^{\infty} \frac{dx}{x(\ln x)^2} = 1/\ln N \le 1/10 \Longleftrightarrow N \ge e^{10} = 22026.46579\ldots,$$

so that $N \ge 22027$ suffices.

(b) How many terms N of the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(\ln n)}$ can we use to guarantee the remainder

$$R_N = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(\ln n)} - \sum_{n=2}^{N} \frac{(-1)^n}{\ln(\ln n)}$$

is less than 0.1?

The series is convergent by the alternating series test since, $b_n = 1/\ln(\ln n) > 0$, $\lim_{n\to\infty} b_n = 0$, and

$$b_n = \frac{1}{\ln(\ln(n))} \ge \frac{1}{\ln(\ln(n+1))} \text{(clear)}.$$

Hence the remainder satisfies $|R_N| \leq b_{N+1} = 1/\ln(\ln(N+1))$. So the absolute value of the remainder $|R_N|$ will be less than 0.1 if

$$\frac{1}{\ln(\ln(N+1))} \leq \frac{1}{10} \iff e^{e^{10}} \leq N+1,$$

so any $N \geq e^{e^{10}} - 1 \approx 9.38 \times 10^{9565}$ will suffice (that's a lot of terms...).