MATH 2300-015 QUIZ 7 Name: ____________________________

Determine whether the following series converge or diverge (circle your answer). Indicate your reasoning (divergence test, integral test, direct or limit comparison test, etc.).

1. $\displaystyle\sum_{j=1}^{\infty} 4^{1/j}$

   The series diverges because the terms do not approach zero (divergence test):

   $$\lim_{j\to\infty} 4^{1/j} = 4^{\lim_{j\to\infty} 1/j} = 4^0 = 1.$$

2. $\displaystyle\sum_{k=0}^{\infty} \frac{4^k}{5^k - k^2}$

   The series converges, comparing in the limit to the convergent geometric series $\sum_k (4/5)^k$:

   $$\lim_{k\to\infty} \frac{(4/5)^k}{4^k/(5^k - k^2)} = \lim_{k\to\infty} \frac{5^k - k^2}{5^k} = \lim_{k\to\infty} \frac{1}{1 - k^2/5^k} = 1$$

   since $\lim_{k\to\infty} k^2/5^k = 0$ (using L'Hôpitals's rule for instance).

3. $\displaystyle\sum_{l=2}^{\infty} \frac{l}{\sqrt{l^3 - l}}$

   The series diverges, comparing directly to the divergent p-series $\sum_l 1/\sqrt{l}$:

   $$\frac{l}{\sqrt{l^3 - l}} \le \frac{l}{\sqrt{l^3}} = \frac{1}{\sqrt{l}}.$$

4. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2n}{3n+5}\right)^n$

   The series converges, comparing directly to the convergent geometric series $\sum_n (2/3)^n$:

   $$\left(\frac{2n}{3n+5}\right)^n \le \left(\frac{2n}{3n}\right)^n = (2/3)^n.$$

5. $\displaystyle\sum_{m=0}^{\infty} \cos(e^{-m})$

   The series diverges because the terms do not approach zero (divergence test):

   $$\lim_{m\to\infty} \cos(e^{-m}) = \cos(\lim_{m\to\infty} e^{-m}) = \cos(0) = 1.$$

6. $\displaystyle\sum_{j=1}^{\infty} \frac{1}{j^{1+\sqrt{j}}}$

   The series converges, comparing directly to the convergent p-series $\sum_j 1/j^2$:

   $$1 + \sqrt{j} \ge 1 + 1 = 2 \Rightarrow \frac{1}{j^{1+\sqrt{j}}} \le \frac{1}{j^2}:$$

7. $\displaystyle\sum_{\nu=1}^{\infty} \frac{2+\sin\nu}{\sqrt{\nu}}$

The series diverges, comparing directly to $\sum_\nu 1/\sqrt{\nu}$:

$$1 \le 2+\sin\nu \Rightarrow \frac{2+\sin\nu}{\sqrt{\nu}} \ge \frac{1}{\sqrt{\nu}}.$$

8. $\displaystyle\sum_{n=0}^{\infty} ne^{-n^2}$

The series converges by the integral test, $ne^{-n^2} = f(n)$ where $f(x) = xe^{-x^2}$. The function $f(x)$ is clearly positive and continuous, and it is decreasing

$$f'(x) = x(-2xe^{-x^2}) + e^{-x^2} = e^{-x^2}(1-2x) < 0 \text{ for } x > 1/2.$$

The associated improper integral converges (with $u = x^2$)

$$\int_1^\infty xe^{-x^2}dx = \frac{1}{2}\int_1^\infty e^{-u}du = 1/2.$$

9. $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$

The series converges by direct comparison to the convergent p-series $\sum_k 1/k^{3/2}$:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0 \Rightarrow \frac{\ln k}{k^2} \le \frac{\sqrt{k}}{k^2} = \frac{1}{k^{3/2}} \text{ for } k \text{ large }.$$

Or, one could use the integral test, since $\frac{\ln k}{k^2} = f(k)$ where $f(x) = \ln x/x^2$ is positive, continuous, and decreasing

$$f'(x) = -2x^{-3}\ln x + x^{-2}(1/x) = -2\frac{1+\ln x}{x^3} < 0 \text{ for } x > 1/e.$$

We have (with $u = \ln x$)

$$\int_1^\infty \frac{\ln x}{x^2}dx = \int_0^\infty ue^{-u}du = -ue^{-u}\Big|_0^\infty + \int_0^\infty e^{-u} = 1.$$

10. $\displaystyle\sum_{m=1}^{\infty} \sin(1/m)$

The series converges, comparing in the limit to the divergent harmonic series $\sum_m 1/m$:

$$\lim_{m\to\infty} \frac{\sin(1/m)}{1/m} = 1.$$