MATH 2300-015 QUIZ 6 Name: ____________________

Due Tuesday, October 10th at the beginning of class. Write your solutions on separate paper.

SHOW YOUR WORK!

Determine whether the sequence converges or diverges. If it converges, find its limit.

1. $a_n = \dfrac{e^n + e^{-n}}{e^{2n} - 1}$

   We have (dividing numerator and denominator by e^{2n})

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^{-n} + e^{-3n}}{1 - e^{-2n}} = \frac{0+0}{1+0} = 0.$$

2. $b_n = \ln(2n^2 + 1) - \ln(n^2 + 1)$

   We have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \ln\left(\frac{2n^2+1}{n^2+1}\right) = \lim_{n\to\infty} \ln\left(\frac{2+1/n^2}{1+1/n^2}\right) = \ln\left(\frac{2+0}{1+0}\right) = \ln 2.$$

3. $c_n = \sqrt[n]{2^n + 3^n}$

   We have

$$c_n = (2^n + 3^n)^{1/n} = [3^n((2/3)^n + 1)]^{1/n} = 3\left(1 + \left(\frac{2}{3}\right)^n\right)^{1/n}.$$

   Let $0 < r < 1$ (for instance $r = 2/3$). Then

$$\lim_{x\to\infty} (1 + r^x)^{1/x} = \lim_{x\to\infty} e^{\frac{1}{x}\ln(1+r^x)} = e^{\lim_{x\to\infty} \frac{\ln(1+r^x)}{x}}$$

   and (using L'Hôpital's rule)

$$\lim_{x\to\infty} \frac{\ln(1+r^x)}{x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{r^x \ln r}{1 + r^x} = \lim_{x\to\infty} \frac{\ln r}{1 + r^{-x}} = 0$$

   since $r^{-x} \to \infty$ as $x \to \infty$. Hence

$$\lim_{n\to\infty} c_n = 3e^0 = 3.$$

   Here is another argument. We have

$$3 = (0 + 3^n)^{1/n} < c_n < (3^n + 3^n)^{1/n} = 3 \cdot 2^{1/n}.$$

   Also, $\lim_{n\to\infty} 2^{1/n} = 1$ since

$$\lim_{n\to\infty} \frac{\ln 2}{n} = 0.$$

   Hence

$$3 \le \lim_{n\to\infty} c_n \le 3 \text{ and } \lim_{n\to\infty} c_n = 3.$$

4. $d_n = \dfrac{\sin(n)\ln n}{n}$

   Since $|\sin(x)| \le 1$ for any $x \in \mathbb{R}$, we have

$$-\frac{\ln n}{n} \le d_n \le \frac{\ln n}{n}.$$

   We also have (using L'Hôpital's rule)

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1}{x} = 0,$$

   so that

$$0 = \lim_{n\to\infty} -\frac{\ln n}{n} \le \lim_{n\to\infty} d_n \le \lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

   Hence $\lim\limits_{n\to\infty} d_n = 0$.

Determine whether the series is convergent or divergent. If it converges find the sum.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$

   Using partial fractions, we have

$$\frac{2}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1}.$$

   The Nth partial sum of the series is

$$\sum_{n=2}^{N} \left(\frac{1}{n-1} - \frac{1}{n+1}\right) = (1-1/3) + (1/2-1/4) + (1/3-1/5) + \cdots + (1/(N-1) - 1/(N+1))$$
$$= 1 + 1/2 - 1/N - 1/(N+1),$$

   noting the cancelation (or "telescoping") in the sum. Taking the limit as $N \to \infty$ we get

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{N\to\infty} 1 + 1/2 - 1/N - 1/(N+1) = 3/2.$$

   [One could determine that this series converges using the integral test or limit comparison test, comparing to $\sum_n \frac{1}{n^2}$, but that won't tell you the value of the series.]

2. $\displaystyle\sum_{k=1}^{\infty} \ln\left(1+\frac{1}{k}\right)$

   This is another telescoping sum. We have

$$\ln(1+1/k) = \ln\left(\frac{k+1}{k}\right) = \ln(k+1) - \ln k,$$

   so the Nth partial sum of the series is

$$\sum_{k=1}^{N} \ln\left(1+\frac{1}{k}\right) = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(N+1) - \ln N) = \ln(N+1)$$

   which diverges as $N \to \infty$. Hence the series diverges. Note that the series diverges even though $\ln(1+1/k) \to \ln 1 = 0$ as $k \to \infty$. [One could determine that this series diverges using the integral test or the limit comparison test, comparing to $\sum_k \frac{1}{k}$.]

3. $\displaystyle\sum_{m=1}^{\infty} \frac{m(m+2)}{(m+3)^2}$

The series diverges because the terms do not approach zero,

$$\lim_{m\to\infty} \frac{m(m+2)}{(m+3)^2} = 1.$$

4. $\displaystyle\sum_{j=1}^{\infty} [(0.8)^{j-1} - (0.3)^j]$

This a sum of two convergent geometric series:

$$\sum_{j=1}^{\infty} [(0.8)^{j-1} - (0.3)^j] = \sum_{n=0}^{\infty} (0.8)^n - \sum_{j=1}^{\infty} (0.3)^j = \frac{1}{1-0.8} - \frac{0.3}{1-0.3} = 5 - 3/7 = 32/7.$$

[Note the reindexing $n = j - 1$ after the first equality.]

Try these more challenging problems from the text.

1. [Section 8.1, problem 55]

   [I will be using the "principle of mathematical induction" in what follows. If you want to show a proposition P holds for all natural numbers $n = 0, 1, 2, 3, \ldots$, one can proceed as follows:

   - prove P is true for $n = 0$ (the "base case"),
   - prove that if P is true for n, then it is true for $n + 1$ (the "inductive step").

   From this it follows that P is true for $n = 0$ by the base case, then for $n = 1$ by the induction step since P is true for $n = 0$, then for $n = 2$ by the inductive step since P is true for $n = 1$, etc., and the proposition holds for all $n \geq 0$.]

   The sequence is increasing by induction since (base case)

   $$a_2 = 2 > 1 = a_1$$

   and (inductive step)

   $$0 < a_n < a_{n+1} \Rightarrow 1/a_n > 1/a_{n+1} \Rightarrow 3 - 1/a_n < 3 - 1/a_{n+1} \Rightarrow a_{n+1} < a_{n+2}.$$

   The sequence is bounded above by 3 since we are subtracting a positive number $1/a_{n-1}$ from 3. The bounded increasing sequence a_n thus has a limit L. Taking limits on both sides of $a_{n+1} = 3 - 1/a_n$ gives

   $$L = 3 - 1/L,\ L^2 - 3L + 1 = 0,\ L = \frac{3 \pm \sqrt{5}}{2},$$

   and $L = \frac{3+\sqrt{5}}{2}$ since $L > a_1 = 1$.

2. [Section 8.2, problem 58]

   Each red line segment in the figure is the hypothenuse of a right triangle with angle θ. Let the sequence of lengths be $(l_n)_{n=1}^{\infty}$. Then

   $$\begin{aligned} \sin\theta &= l_1/b, \\ \sin\theta &= l_2/l_1, \\ &\cdots \\ \sin\theta &= l_{n+1}/l_n, \end{aligned}$$

   i.e. $l_n = b\sin^n\theta$. Hence the total length of the red zig-zag is

   $$\sum_{n=1}^{\infty} b\sin^n\theta = \frac{b\sin\theta}{1-\sin\theta}.$$

3. [Section 8.2, problem 68] Answer: $11\pi/96$. Hint: If $(r_n)_{n=0}^{\infty}$ is the sequence of radii of the successively smaller circles, then the ratio r_{n+1}/r_n is constant.

The total area is

$$A = \pi r_0^2 + 3\sum_{n=1}^{\infty} \pi r_n^2,$$

adding the area of the middle circle and the areas of the three sequence of circles going towards the vertices. So we must determine the radii r_n.

Let $x = r_0$ be the length of the segment from the center of the triangle to one of the sides and perpendicular to that side, and let y be length of the segment from the center of the triangle to one of the vertices. Then

$$x + y = \sqrt{3}/2,\ x = y/2 \Rightarrow x = \frac{1}{2\sqrt{3}},\ y = 2x = \frac{1}{\sqrt{3}}.$$

Drawing a line parallel to the base of the triangle tangent to the top of the largest circle creates another equilateral triangle containing the rest of the circles going towards the top vertex. The smaller triangle is 1/3 the size of the larger triangle since the height of the smaller triangle is x and the height of the large triangle is $3x$ (see figure). Hence

$$r_1 = r_0/3,\ r_2 = r_1/3,\ \ldots,\ r_{n+1} = r_n/3 \Rightarrow r_n = \frac{r_0}{3^n}.$$

Hence the total area is

$$\begin{aligned} A &= \pi r_0^2 + 3\sum_{n=1}^{\infty} \pi r_n^2 = \pi r_0^2 + 3\sum_{n=1}^{\infty} \pi (r_0/3^n)^2 = \pi r_0^2 + 3\pi r_0^2 \sum_{n=1}^{\infty} (1/9)^n \\ &= \pi r_0^2 \left(1 + 3\frac{1/9}{1-1/9}\right) = 11\pi/96. \end{aligned}$$