MATH 2300-015 QUIZ 5 Name: ________________________

Due Tuesday, October 3rd at the beginning of class. Please use additional paper as necessary to submit CLEAR and COMPLETE solutions.

1. [Section 6.6, exer. 13] A cable that weighs $2\frac{\text{lb}}{\text{ft}}$ is used to lift 800 lb of coal up a mine shaft 500 ft deep. Find the work done.

2. [Section 6.6, exer. 17] An aquarium 2 m long, 1 m wide, and 1 m deep is full of water. Find the work needed to pump half of the water out of the aquarium. (Use the fact that the density of water is $1000\frac{\text{kg}}{\text{m}^3}$ and that the acceleration due to gravity near the earth's surface is $9.8\frac{\text{m}}{\text{s}^2}$.)

3. Find the centroid $(\bar{x}, \bar{y})$ of the region

$$\left\{(x, y) : 1 \leq x < \infty, 0 \leq y \leq \frac{1}{x^3}\right\},$$

i.e. the region bounded by $y = 0$ and $y = \frac{1}{x^3}$ for $1 \leq x < \infty$. [Note that the integrals for the moments and area are improper. Even though the region is unbounded in the x-direction, its centroid is still finite.]

Solutions

1. The work done lifting just the coal is (800lb)(500ft) = 400,000 ft-lb. The work done lifting a small length Δx of cable up x ft is $(2\frac{\text{lb}}{\text{ft}})(\Delta x \text{ ft})(x \text{ ft}) = 2x\Delta x$ ft-lb. Summing this over the length of the cable (500 ft) and taking a limit gives the integral

$$W = \int_0^{500} 2x\, dx = 250,000 \text{ ft-lb}.$$

Hence the total work done is 650,000 ft-lb.

2. To lift a rectangular slab (of length 2 m, wideth 1 m, and height Δx m) at depth x m, the work required is

$$(2\Delta x \text{ m}^3)\left(1000\frac{\text{kg}}{\text{m}^3}\right)\left(9.8\frac{\text{m}}{\text{s}^2}\right)(x \text{ m}) = 19,600x\Delta x \text{ J}.$$

Summing this up over half the depth ($0 \leq x \leq 0.5$ m) and taking a limit gives the integral

$$\int_0^{1/2} 19,600x\, dx = 2,450 \text{ J}.$$

3. We have

$$M_x = \int_1^\infty \frac{1}{2}\left(\frac{1}{x^3}\right)^2 dx = \lim_{t\to\infty} -\frac{1}{10x^5}\Big|_1^t = \frac{1}{10},$$

$$M_y = \int_1^\infty x\frac{1}{x^3} dx = \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t = 1,$$

$$A = \int_1^\infty \frac{1}{x^3} dx = \lim_{t\to\infty} -\frac{1}{2x^2}\Big|_1^t = \frac{1}{2}.$$

Hence the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{A}, \frac{M_x}{A} \right) = \left(2, \frac{1}{5} \right).$$

Note that $\frac{1}{5} < \frac{1}{2^3}$ so that the centroid does not lie in the region. The center of mass of an object need not lie within the object, e.g. boomerangs, horseshoes, bowls, etc.