MATH 2300-015 QUIZ 4 Name: ____________________________

Due Monday, September 25th at the beginning of class. Please use additional paper as necessary to submit CLEAR and COMPLETE solutions.

1. Find the volume of the solid obtained by rotating the region bounded by the curves

$$y = 0, \; y = (1-x^2)^{3/4}$$

around

(a) the x-axis:
The cross-sections perpendicular to the x-axis are disks parameterized by x of area $\pi r^2 = \pi y^2 = \pi((1-x^2)^{3/4})^2$. Hence the volume is

$$\int_{-1}^{1} \pi((1-x^2)^{3/4})^2 dx = \pi \int_{-1}^{1} (1-x^2)^{3/2} dx = \pi \int_{-\pi/2}^{\pi/2} \cos^4 \theta d\theta$$
$$= \frac{\pi}{2} \int_0^{\pi/2} \left(1 + 2\cos(2\theta) + \frac{1+\cos(4\theta)}{2}\right) d\theta = \frac{\pi}{2}(3\theta/2 + \sin(2\theta) + \sin(4\theta)/8)\Big|_0^{\pi/2}$$
$$= \frac{3\pi^2}{8} \approx 3.7011.$$

(b) the line $x = 1$:
Using the method of cylindrical shells, we obtain a volume of

$$\int_{-1}^{1} 2\pi(1-x)(1-x^2)^{3/4} dx = 2\pi \int_{-1}^{1} (1-x^2)^{3/4} dx = ???$$

The integrand of the last integral doesn't have a "nice" antiderivative ☹. (Although later in the course we will see how to expand the integrand as a power series, which we could integrate term by term, and give the resulting volume as an infinite series.)

2. Find the volume of the solid obtained by rotating the region bounded by the curves

$$y = 0, \; x = 0, \; y = xe^{-x}$$

around (note that these are improper integrals since the region extends infinitely along the x-axis)

(a) the y-axis:
Using the method of cylindrical shells, the volume is

$$\int_0^{\infty} 2\pi x(xe^{-x}) dx = 2\pi \lim_{t\to\infty} \int_0^t x^2 e^{-x} dx = 2\pi \lim_{t\to\infty} -x^2 e^{-x}\Big|_0^t + 2\int_0^t xe^{-x} dx$$
$$= 2\pi \lim_{t\to\infty} -x^2e^{-x} - 2xe^{-x}\Big|_0^t + 2\int_0^t e^{-x} = 2\pi \lim_{t\to\infty} -x^2e^{-x} - 2xe^{-x} - 2e^{-x}\Big|_0^t$$
$$= 4\pi - 2\pi \lim_{t\to\infty} e^{-t}(t^2 + 2t + 2) = 4\pi$$

using, say, L'Hôpital's rule to compute the limit (exponential growth beats polynomial growth evey time, i.e. $\lim_{x\to\infty} x^n/e^x = 0$ for any n).

(b) the line $y = -1$:

The cross-sections perpendicular to the axis $y = -1$ are "washers" and the volume is

$$\begin{aligned}
\int_0^\infty \pi[(xe^{-x}+1)^2 - (1)^2]dx &= \pi \lim_{t\to\infty} \int_0^t (x^2e^{-2x} + 2xe^{-x})dx \\
&= \pi \lim_{t\to\infty} \left[-x^2e^{-2x}/2\Big|_0^t + \int_0^t xe^{-2x}dx - 2xe^{-x}\Big|_0^t + 2\int_0^t e^{-x}dx \right] \\
&= \pi \lim_{t\to\infty} \left[-x^2e^{-2x}/2 - 2xe^{-x} - 2e^{-x} - xe^{-2x}/2\Big|_0^t + \frac{1}{2}\int_0^t e^{-2x}dx \right] \\
&= \pi \lim_{t\to\infty} \left[-x^2e^{-2x}/2 - 2xe^{-x} - 2e^{-x} - xe^{-2x}/2 - e^{-2x}/4\Big|_0^t \right] \\
&= \frac{9\pi}{4} - \lim_{t\to\infty} \left(\frac{e^{-2x}}{4}(2x^2+2x+1) + 2e^{-x}(x+1) \right) \\
&= \frac{9\pi}{4} \approx 7.0686,
\end{aligned}$$

using L'Hôpital's rule for the limit in the last step.

3. Find the arclength of the curve $y = \ln x$ for $x \in [1, \sqrt{3}]$.

We need to integrate

$$\int_1^{\sqrt{3}} \sqrt{1+(1/x)^2}dx = \int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x}dx.$$

With the substitution $x = \tan\theta$, $dx = \sec^2\theta d\theta$, this becomes

$$\int_{\pi/4}^{\pi/3} \frac{\sec\theta}{\tan\theta}\sec^2\theta d\theta = \int_{\pi/4}^{\pi/3} \frac{d\theta}{\sin\theta\cos^2\theta} = \int_{\pi/4}^{\pi/3} \frac{\sin\theta d\theta}{\sin^2\theta\cos^2\theta}.$$

With $u = \cos\theta$, $du = -\sin\theta d\theta$, we get

$$\int_{1/\sqrt{2}}^{1/2} \frac{d\theta}{u^2(u^2-1)} = \int_{1/\sqrt{2}}^{1/2} \frac{d\theta}{u^2(u-1)(u+1)}.$$

The partial fraction decomposition of the integrand is

$$\begin{aligned}
&\frac{1}{u^2(u-1)(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-1} + \frac{D}{u+1} \\
&1 = (A+C+D)u^3 + (B+C-D)u^2 + (-A)u + (-B) \\
&A = 0,\ B = -1,\ C = 1/2,\ D = -1/2.
\end{aligned}$$

Hence the integral becomes

$$\begin{aligned}
\int_{1/\sqrt{2}}^{1/2} \left(\frac{-1}{u^2} + \frac{1/2}{u-1} + \frac{-1/2}{u+1} \right) du &= \left(\frac{1}{u} + \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| \right)\Big|_{1/\sqrt{2}}^{1/2} \\
&= 2 - \sqrt{2} + \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{3(\sqrt{2}-1)} \right) = 0.91785\ldots.
\end{aligned}$$

Alternatively, the trig integral above can be written as

$$\int_{\pi/4}^{\pi/3} \frac{d\theta}{\sin\theta\cos^2\theta} = \int_{\pi/4}^{\pi/3} \left(\csc\theta + \frac{\sin\theta}{\cos^2\theta}\right) d\theta = -\ln(\csc\theta + \cot\theta)\Big|_{\pi/4}^{\pi/3} - \int_{1/\sqrt{2}}^{1/2} \frac{du}{u^2}$$

bypassing the partial fraction decomposition.

Or, yet another way to integrate the original integral is with the substitution

$$u = \sqrt{1+x^2},\ x = \sqrt{u^2-1}$$

$$du = \frac{xdx}{\sqrt{1+x^2}} = \frac{\sqrt{u^2-1}}{u}dx,\ dx = \frac{udu}{\sqrt{u^2-1}}$$

so that the integral becomes

$$\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x}dx = \int_{\sqrt{2}}^2 \frac{u^2}{u^2-1}du = \int_{\sqrt{2}}^2 \left(1 + \frac{1}{(u-1)(u+1)}\right) du$$

which can be done with partial fractions.