MATH 2300-015 QUIZ 3 Name: ____________________________

Due Tuesday, September 19th at the beginning of class. Please use additional paper as necessary to submit CLEAR and COMPLETE solutions.

1. For which values of p does $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)^p}$ converge/diverge? Find the value of the improper integral when it is convergent.

If $p \neq 1$ we have

$$\begin{aligned}\int_e^\infty \frac{dx}{x(\ln x)^p} &= \lim_{t\to\infty}\int_e^t \frac{dx}{x(\ln x)^p} = \lim_{t\to\infty}\int_1^{\ln t}\frac{du}{u^p} = \lim_{t\to\infty}\frac{u^{1-p}}{1-p}\Big|_1^{\ln t}\\ &= \lim_{t\to\infty}\frac{(\ln t)^{1-p}}{1-p} - \frac{1}{1-p}\end{aligned}$$

which is $\frac{1}{p-1}$ if $p>1$ and ∞ if $p<1$. For $p=1$ we get $\lim\limits_{t\to\infty}\ln(\ln t)=\infty$ and the integral diverges as well. In summary

$$\int_e^\infty \frac{dx}{x(\ln x)^p} = \begin{cases}\infty & p\le 1\\ \frac{1}{p-1} & p>1\end{cases}.$$

2. For what values of p does the improper integral $\displaystyle\int_0^1 \frac{dx}{x^p}$ converge?

For $p\neq 1$ we have

$$\begin{aligned}\int_0^1 \frac{dx}{x^p} &= \lim_{t\to 0^+}\int_t^1\frac{dx}{x^p} = \lim_{t\to 0^+}\frac{x^{1-p}}{1-p}\Big|_t^1\\ &= \frac{1}{1-p} - \lim_{t\to 0^+}\frac{t^{1-p}}{1-p}\end{aligned}$$

which is $\frac{1}{1-p}$ if $p<1$ and ∞ if $p>1$. For $p=1$ we get $\lim\limits_{t\to 0^+} -\ln t = \infty$ and the integral diverges as well. In summary

$$\int_0^1 \frac{dx}{x^p} = \begin{cases}\infty & p\ge 1\\ \frac{1}{1-p} & p<1\end{cases}.$$

3. First, show that $\displaystyle\int_0^\infty \frac{dx}{x^3+1}$ converges by comparison. Second, find the value of the improper integral. (You should get $\frac{2\pi}{3\sqrt{3}}$).

For instance, we can compare $\frac{1}{1+x^3}\le\frac{1}{x^3}$ on $[1,\infty)$ so that

$$\int_0^\infty \frac{dx}{1+x^3} = \int_0^1\frac{dx}{1+x^3} + \int_1^\infty\frac{dx}{1+x^3} \le \int_0^1\frac{dx}{1+x^3} + \int_1^\infty \frac{dx}{x^3} < \infty.$$

As for the actual value, we use partial fractions:

$$\frac{1}{1+x^3} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1},$$
$$1 = (A+B)x^2 + (-A+B+C)x + (A+C),$$
$$A = \frac{1}{3},\ B = -\frac{1}{3},\ C = \frac{2}{3}.$$

Hence the improper integral is

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{3}\int_0^t \left(\frac{1}{x+1} + \frac{-x+2}{x^2-x+1}\right) dx \\
&= \lim_{t\to\infty} \frac{1}{3}\ln|t+1| + \frac{1}{6}\int_0^t \frac{-2x+1}{x^2-x+1}dx + \frac{1}{2}\int_0^t \frac{dx}{x^2-x+1} \\
&= \lim_{t\to\infty} \frac{1}{3}\ln|t+1| - \frac{1}{6}\ln|t^2-t+1| + \frac{1}{2}\int_0^t \frac{dx}{(x-1/2)^2+3/4} \\
&= \lim_{t\to\infty} \frac{1}{3}\ln|t+1| - \frac{1}{6}\ln|t^2-t+1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2t-1}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}}\arctan(-1/\sqrt{3}) \\
&= \frac{\pi}{2\sqrt{3}} + \frac{\pi}{6\sqrt{3}} + \lim_{t\to\infty}\frac{1}{3}\ln\left|\frac{t+1}{\sqrt{t^2-t+1}}\right| = \frac{2\pi}{3\sqrt{3}}.
\end{aligned}$$

In the last line we are using the limit

$$\lim_{t\to\infty}\frac{t+1}{\sqrt{t^2-t+1}} = \lim_{t\to\infty}\frac{1+1/t}{\sqrt{1-1/t+1/t^2}} = 1.$$

4. Find the value of C for which the following improper integral converges and evaluate the integral for this value of C:

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right) dx.$$

[Note that the integral of each summand separately is divergent, but the right choice of C gives "cancellation" and a convergent integral.]

The integral is (with $x = 2\tan\theta$ integrating the first summand)

$$\begin{aligned}
&\lim_{t\to\infty}\int_0^{\arctan(t/2)} \sec\theta\ d\theta - C\ln|t+2| + C\ln 2 \\
&= \lim_{t\to\infty} \ln|\sec(\arctan(t/2)) + \tan(\arctan(t/2))| - C\ln|t+2| + C\ln 2 \\
&= \lim_{t\to\infty} \ln|\frac{1}{2}\sqrt{t^2+4} + t/2| - C\ln|t+2| + C\ln 2 = \lim_{t\to\infty}\ln\left|\frac{\sqrt{t^2+4}+t}{2(t+2)^C}\right| + C\ln 2.
\end{aligned}$$

Now we see that $C = 1$ is the only possibility, else $\frac{\sqrt{t^2+4}+t}{2(t+2)^C}$ goes to 0 or ∞ as $t\to\infty$ and the logarithm will diverge. For $C = 1$, the value of the integral is $\ln 2$.

5. Recall Simpson's rule for approximating a definite integral:

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1)} + f(x_n)),$$

where

$$\Delta x = \frac{b-a}{n},\ x_i = a + i\Delta x,\ n \text{ even}.$$

(This estimate is obtained by approximating f piecewise on $[x_{2k}, x_{2k+2}]$ by the unique quadratic through the three points $(x_{2k}, f(x_{2k}))$, $(x_{2k+1}, f(x_{2k+1}))$, and $(x_{2k+2}, f(x_{2k+2}))$. Cf. pp. 406-410 of the text.)

A bound for the error

$$E_S := \int_a^b f(x)dx - S_n$$

is given by

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

where K is any bound for the fourth derivative of f on the interval $[a, b]$,

$$K \geq |f^{(4)}(x)|,\ x \in [a, b].$$

Using the above, find a value of n large enough to guarantee $|E_S| < 10^{-6}$ when approximating the integral

$$\int_0^1 e^{-x^2} dx.$$

[Note: A computer gives $f^{(4)}(x) = 4e^{-x^2}(4x^4 - 12x^2 + 3)$ when $f(x) = e^{-x^2}$. Use the methods of Calc 1 to find the max/min of the fourth derivative on the interval $[0, 1]$, and use this to determine a value for K.]

First we bound $f^{(4)}(x)$ on $[0, 1]$ where $f(x) = e^{-x^2}$. We have

$$\begin{aligned}
f' &= -2xe^{-x^2} \\
f'' &= -2x(-2xe^{-x^2}) - 2e^{-x^2} = (4x^2 - 2)e^{-x^2} \\
f^{(3)} &= (4x^2 - 2)(-2xe^{-x^2}) + e^{-x^2}(8x) = e^{-x^2}(-8x^3 + 12x) \\
f^{(4)} &= -2xe^{-x^2}(-8x^3 + 12x) + e^{-x^2}(-24x^2 + 12) = 4e^{-x^2}(4x^4 - 12x^2 + 3) \\
f^{(5)} &= -2xe^{-x^2}(16x^4 - 48x^2 + 12) + e^{-x^2}(64x^3 - 96x) = -8xe^{-x^2}(4x^4 - 20x^2 + 15).
\end{aligned}$$

The fifth derivative has zeros at $x = 0$ and

$$4x^4 - 20x^2 + 15 = 0 \Leftrightarrow x^2 = \frac{20 \pm \sqrt{20^2 - 4 \cdot 4 \cdot 15}}{8} \Leftrightarrow x = \pm\sqrt{\frac{5 \pm \sqrt{10}}{2}} = \pm 2.021\ldots, \pm 0.958\ldots$$

We find the absolute max/min of $f^{(4)}$ on $[0, 1]$ by testing the value of $f^{(4)}$ at critical points and the endpoints

$$f^{(4)}(0) = 12,\ f^{(4)}(1) = -20/e = -7.357\ldots\ f^{(4)}\left(\sqrt{\frac{5 - \sqrt{10}}{2}}\right) = -7.419\ldots.$$

Hence we can take $|f^{(4)}(x)| \leq 12 = K$ on $[0,1]$. With this value of K and $a = 0, b = 1$ the error bound is

$$|E_S| \leq \frac{K(b-a)^5}{180n^4} = \frac{1}{15n^4}.$$

Now, if we want $|E_S| \leq 10^{-6}$, we can take the smallest even n satisfying

$$\frac{1}{15n^4} \leq 10^{-6}.$$

Solving for n gives

$$n \geq (10^6/15)^{1/4} = 16.06\ldots.$$

Therefore we take $n = 18$ (the smallest even integer greater than $16.06\ldots$).