

Due Tuesday, September 12th at the beginning of class. Please use additional paper as necessary to submit CLEAR and COMPLETE solutions.

Use the partial fraction decomposition to integrate these rational functions.

1.  $\int \frac{x^2 - x + 5}{x^2 + x - 6} dx$ . The integrand is

$$\frac{x^2 - x + 5}{x^2 + x - 6} = 1 + \frac{-2x + 11}{x^2 + x - 6} = 1 + \frac{-2x + 11}{(x + 3)(x - 2)},$$

so we want the partial fraction decomposition of  $\frac{-2x+11}{(x+3)(x-2)}$ ,

$$\begin{aligned} \frac{-2x + 11}{(x + 3)(x - 2)} &= \frac{A}{x + 3} + \frac{B}{x - 2} \Rightarrow -2x + 11 = A(x - 2) + B(x + 3) = (A + B)x - 2A + 3B \\ \Rightarrow -2 &= A + B, \quad 11 = -2A + 3B \Rightarrow A = -17/5, \quad B = 7/5. \end{aligned}$$

Hence the integral is

$$\int dx - \frac{17}{5} \int \frac{dx}{x + 3} + \frac{7}{5} \int \frac{dx}{x - 2} = x - \frac{17}{5} \ln |x + 3| + \frac{7}{5} \ln |x - 2|.$$

2.  $\int \frac{y^2 + 3}{y^3 - 3y^2 + 3y - 1} dy$ . The integrand is

$$\frac{y^2 + 3}{y^3 - 3y^2 + 3y - 1} = \frac{y^2 + 3}{(y - 1)^3} = \frac{A}{y - 1} + \frac{B}{(y - 1)^2} + \frac{C}{(y - 1)^3},$$

and we obtain

$$\begin{aligned} y^2 + 3 &= A(y - 1)^2 + B(y - 1) + C = Ay^2 + (B - 2A)y + A - B + C \\ \Rightarrow A &= 1, \quad B - 2A = 0, \quad A - B + C = 3 \Rightarrow A = 1, \quad B = 2, \quad C = 4. \end{aligned}$$

Hence the integral is

$$\int \frac{dy}{y - 1} + \int \frac{2dy}{(y - 1)^2} + \int \frac{4dy}{(y - 1)^3} = \ln |y - 1| - 2(y - 1)^{-1} - 2(y - 1)^{-2}.$$

3.  $\int \frac{-z^3 - 26z^2 - 28z - 120}{z^4 - 16} dz$ . The integrand is

$$\frac{-z^3 - 26z^2 - 28z - 120}{z^4 - 16} = \frac{-z^3 - 26z^2 - 28z - 120}{(z^2 + 4)(z + 2)(z - 2)} = \frac{Az + B}{z^2 + 4} + \frac{C}{z + 2} + \frac{D}{z - 2}.$$

We must solve

$$\begin{aligned} -z^3 - 26z^2 - 28z - 120 &= (Az + B)(z + 2)(z - 2) + C(z - 2)(z^2 + 4) + D(z + 2)(z^2 + 4) \\ &= (A + C + D)z^3 + (B - 2C + 2D)z^2 + (-4A + 4C + 4D)z + (-4B - 8C + 8D). \end{aligned}$$

More compactly, we can reduce

$$\begin{pmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & -2 & 2 & -26 \\ -4 & 0 & 4 & 4 & -28 \\ 0 & -4 & -8 & 8 & -120 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -9 \end{pmatrix}$$

to get  $A = 3$ ,  $B = 2$ ,  $C = 5$ ,  $D = -9$ . Hence the integral is

$$\begin{aligned} \int \frac{3z+2}{z^2+4} dz + \int \frac{5dz}{z+2} + \int \frac{-9dz}{z-2} &= \frac{3}{2} \int \frac{2z}{z^2+4} dz + 2 \int \frac{dz}{z^2+4} + 5 \int \frac{dz}{z+2} - 9 \int \frac{dz}{z-2} \\ &= \frac{3}{2} \ln(z^2+4) + \arctan(z/2) + 5 \ln|z+2| - 9 \ln|z-2|. \end{aligned}$$

4.  $\int \frac{dw}{w^2 - 2w + 10}$ . Completing the square, the denominator is  $(w-1)^2 + 9$  and the integral is

$$\frac{1}{3} \arctan\left(\frac{x-1}{3}\right).$$