MATH 2300-015 QUIZ 2 Name: ____________________________

1. $\displaystyle\int_0^{\pi/3} \sin^2 x \cos^3 x dx$

   With the substitution $u = \sin x$, $du = \cos x$, we have

$$\int_0^{\pi/3} \sin^2 x \cos^3 x dx = \int_0^{\pi/3} \sin^2 x(1-\sin^2 x)\cos x dx = \int_0^{\sqrt{3}/2} u^2(1-u^2)du$$
$$= u^3/3 - u^5/5\Big|_0^{\sqrt{3}/2} = \frac{11\sqrt{3}}{160}.$$

2. $\displaystyle\int \tan^3 x \sec^2 x dx$

   With $u = \tan x$, $du = \sec^2 x dx$, we have

$$\int \tan^3 x \sec^2 x dx = \int u^3 du = u^4/4 + C = \frac{1}{4}\tan^4 x + C.$$

   Or, with $u = \sec x$, $du = \sec x \tan x dx$, we have

$$\int \tan^3 x \sec^2 x dx = \int (\sec^2 x - 1)\sec x(\sec x \tan x)dx = \int u(u^2-1)du = u^4/4 - u^2/2 + C$$
$$= \frac{1}{4}\sec^4 x - \frac{1}{2}\sec^2 x + C.$$

   You can check that the answers agree (up to a constant! which is covered by the" $+C$") with $\tan^2 x + 1 = \sec^2 x$.

3. $\displaystyle\int \cos^2 x \sin^2 x dx$

   There are a few ways to do this. Using $\sin(2x) = 2\sin x \cos x$ we have

$$\int \cos^2 x \sin^2 x dx = \frac{1}{4}\int \sin^2(2x)dx = \frac{1}{4}\int \frac{1-\cos(4x)}{2}dx$$
$$= \frac{1}{8}\left(x - \frac{\sin(4x)}{4}\right) + C.$$

   Using the half-angle identities, $\cos^2 x = \frac{1+\cos(2x)}{2}$, $\sin^2 x = \frac{1-\cos(2x)}{2}$, we have

$$\int \cos^2 x \sin^2 x dx = \frac{1}{4}\int (1-\cos^2(2x))dx = \frac{1}{4}\int \left(1 - \frac{1+\cos(4x)}{2}\right)dx$$
$$= \frac{1}{8}\left(x - \frac{\sin(4x)}{4}\right) + C.$$

   Finally, you could also do

$$\int \cos^2 x \sin^2 x dx = \int \cos^2 x(1-\cos^2 x)dx = \int \frac{1+\cos(2x)}{2}\left(1 - \frac{1+\cos(2x)}{2}\right)dx$$
$$= \frac{1}{4}\int (1-\cos^2(2x))dx = \ldots = \frac{1}{8}\left(x - \frac{\sin(4x)}{4}\right) + C.$$

4. $\displaystyle\int_{\sqrt{2}}^{2} \frac{dt}{t^4\sqrt{t^2-1}}$ With $t = \sec\theta$, $dt = \sec\theta\tan\theta d\theta$, we have (notice the change in bounds!)

$$\int_{\sqrt{2}}^{2} \frac{dt}{t^4\sqrt{t^2-1}} = \int_{\pi/4}^{\pi/3} \frac{\sec\theta\tan\theta d\theta}{\sec^4\theta\sqrt{\tan^2\theta}} = \int_{\pi/4}^{\pi/3} \frac{d\theta}{\sec^3\theta} = \int_{\pi/4}^{\pi/3} \cos^3\theta d\theta.$$

Making the substitution $u = \sin\theta$, $du = \cos\theta d\theta$ (changing bounds again)

$$\int_{\pi/4}^{\pi/3} \cos^3\theta d\theta = \int_{1/\sqrt{2}}^{\sqrt{3}/2} (1-u^2)du = u - u^3/3\Big|_{1/\sqrt{2}}^{\sqrt{3}/2} = \frac{9\sqrt{3}-10\sqrt{2}}{24}.$$

5. $\int y^2\sqrt{1-y^2}dy$

With $y = \sin\theta$, $dy = \cos\theta d\theta$, we have

$$\int y^2\sqrt{1-y^2}dy = \int \sin^2\theta\cos^2\theta d\theta = \frac{1}{8}\left(\theta - \frac{\sin(4\theta)}{4}\right) + C$$

(using the result of problem 3). Switching back to $y = \sin\theta$ we get

$$\frac{1}{8}\left(\theta - \frac{\sin(4\theta)}{4}\right) + C = \frac{1}{8}\theta - \frac{1}{32}(4\sin\theta\cos\theta(1-2\sin^2\theta)) + C$$
$$= \frac{1}{8}\arcsin y + \frac{1}{8}y\sqrt{1-y^2}(2y^2-1) + C.$$