MATH 2300-015 QUIZ 12 Due Friday, December 1st Name: ______________________

1. Solve the following initial value problems.

   (a) $y' + y^2 \sin x = 0,\ y(0) = -1/2$
   Solution. Rearranging, we have

$$\frac{dy}{dx} = -y^2 \sin x,\ \frac{dy}{y^2} = -\sin x dx$$

   so that

$$\begin{aligned} \int \frac{dy}{y^2} &= -\int \sin x dx \\ -\frac{1}{y} &= \cos x + C \\ y &= \frac{-1}{\cos x + C}. \end{aligned}$$

   If $y(0) = -1/2$ then $C = 1$ and the solution to the initial value problem is

$$y(x) = \frac{-1}{1 + \cos x}.$$

   (b) $y' = \dfrac{x^2}{y(1+x^3)},\ y(0) = -1$
   Solution. Separating variables gives

$$y\ dy = \frac{x^2}{1+x^3} dx.$$

   Integrating, we obtain

$$\begin{aligned} \int y\ dy &= \int \frac{x^2}{1+x^3} dx, \\ \frac{y^2}{2} &= \frac{1}{3} \ln|1+x^3| + C, \\ y &= \pm\sqrt{\frac{2}{3} \ln|1+x^3| + C}. \end{aligned}$$

   If $y(0) = -1$, then we must have $C = 1$ and the negative square root,

$$y(x) = -\sqrt{\frac{2}{3} \ln|1+x^3| + 1}$$

2. Suppose $y(x)$ is the solution to the initial value problem

$$y' = x^2 - y^2,\ y(0) = 1.$$

   Use Euler's method (step size 0.1) to approximate $y(0.5)$.

   Solution. The approximation is $y(0.5) \approx 0.674295419$. The relevant data are in the table below, where $y_{n+1} = y_n + (0.1)(x_n^2 - y_n^2)$:

n	x_n	y_n	$x_n^2 - y_n^2$
0	0	1	-1
1	0.1	0.9	-0.8
2	0.2	0.82	-0.6324
3	0.3	0.75676	-0.482685698
4	0.4	0.70849143	-0.34196106
5	0.5	0.674295419	

3. Use the third degree Taylor polynomial (centered at zero) for $f(x) = \ln(1+x)$ to estimate $\ln(2)$ and use Taylor's inequality to give bounds on the error.

   Solution. The first four derivatives of $f(x) = \ln(1+x)$ are

$$f'(x) = \frac{1}{1+x},\ f''(x) = \frac{-1}{(1+x)^2},\ f'''(x) = \frac{2}{(1+x)^3},\ f^{(4)}(x) = \frac{-6}{(1+x)^4}.$$

   The third degree Taylor polynomial centered at zero is

$$T_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}.$$

   To use $T_3(x)$ to approximate $\ln(2)$ we take $x = 1$, $\ln(2) \approx T_3(1) = 1 - 1/2 + 1/3 = 5/6$. A bound for the absolute value of the fourth derivative $f^{(4)}(x)$ on the interval $[0, 1]$ is

$$|f^{(4)}(x)| = \left|\frac{-6}{(1+x)^4}\right| \leq 6 = M$$

   and Taylor's inequality states that

$$|\ln(2) - T_3(1)| = |R_n(1)| \leq \frac{M}{(3+1)!}|1-0|^{3+1} = \frac{1}{4}.$$

   Hence

$$7/12 = 5/6 - 1/4 \leq \ln(2) \leq 5/6 + 1/4 = 13/12.$$

4. Recall Newton's law of cooling: the rate of change in temperature of an object is proportional to the difference in temperature between the object and its surroundings,

$$\frac{dT}{dt} = k(T - T_s),$$

   where $T(t)$ is temperature as a function of time, k is the proportionality constant, and T_s is the constant surrounding temperature.

   Suppose a cup of coffee is 200°F when it is poured and has cooled to 190°F after one minute in a room at 70°F. When will the coffee reach 150°F? What will the temperature of the coffee be after it sits for 30 minutes?

   Solution. Separating, integrating, and solving for T gives

$$T(t) = T_s + Ce^{kt}.$$

   Using the information provided ($T(0) = 200$, $T(1) = 190$, $T_s = 70$) allows us to solve for k and C

$$200 = T(0) = 70 + C \Rightarrow C = 130,\ 190 = T(1) = 70 + 130e^k \Rightarrow k = \ln(12/13).$$

To find when the coffee reaches 150°F, we solve

$$150 = T(t) = 70 + 130e^{kt}, \; t = \frac{1}{k}\ln(8/13) \approx 6.06 \text{ minutes.}$$

After 30 minutes, the temperature is

$$T(30) = 70 + 130e^{30k} \approx 81.8°\text{F}.$$

5. The following variation on the logistic equation models logistic growth with constant harvesting:

$$\frac{dP}{dt} = kP(1 - P/M) - c.$$

For this problem consider the specific instance

$$\frac{dP}{dt} = 0.08P(1 - P/1000) - 15,$$

modeling fish population in a pond where 15 fish per week are caught (time t in weeks).

(a) What are the equilibrium solutions to the differential equation in part (i.e. what are the constant solutions)?

Solution. The equilibrium solutions are the zeros of the right-hand side of the differential equation, $P = 750, 250$.

(b) Find the general solution of the differential equation. [Integrate using partial fractions. You should get $P(t) = \dfrac{750 - 250Ce^{-t/25}}{1 - Ce^{-t/25}}$ where C is an arbitrary constant.]

Solution. Separating variables, multiplying by a constant, and integrating gives

$$\int \frac{dP}{P^2 - 1000P + 187500} = -\int \frac{dt}{12500}.$$

Partial fractions on the left-hand side gives

$$\frac{1}{P^2 - 1000P + 187500} = \frac{1}{(P - 750)(P - 250)} = \frac{1/500}{P - 750} + \frac{-1/500}{P - 250}.$$

Integrating, we obtain

$$\ln\left|\frac{P - 750}{P - 250}\right| = -\frac{t}{25} + C.$$

Exponentiating gives (different C)

$$\frac{P - 750}{P - 250} = Ce^{-t/25}$$

Finally, solving for P gives the general solution

$$P(t) = \frac{750 - 250Ce^{-t/25}}{1 - Ce^{-t/25}}$$

(c) Find and interpret the solutions with initial conditions $P(0) = 200, 300$.

Solution. We need to solve for C in the following

$$200 = \frac{750 - 250C}{1 - C}, \ 300 = \frac{750 - 250C}{1 - C}$$

obtaining $C = 11$ and $C = -9$ respectively. So the two particular solutions are

$$P(t) = \frac{750 - 2750e^{-t/25}}{1 - 11e^{-t/25}}, \ P(t) = \frac{750 + 2250e^{-t/25}}{1 + 9e^{-t/25}}.$$

In the first solution ($P(0) = 200$), the population reaches zero at $t = 25\ln(11/3) \approx 32.48$ weeks, i.e. fishing at a rate of 15 fish/week is unsustainable, while in the second solution ($P(0) = 300$), as $t \to \infty$ the population approaches 750 and the population can sustain this level of fishing.